Conservation laws and generalized isometric embeddings

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Conservation Laws and Generalized Isometric Embeddings

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[1] Si vous pensez que c'est agréable pour un non-mathématicien de lire ce que j'écris, veuillez tourner quelques pages et mettez-vous à lire. Vous comprendrez assez rapidement.

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Introduction (en français)[2]

Ce travail de thèse se situe dans le domaine de la géométrie différentielle et a pour objectif l'étude du problème du plongement isométrique généralisé de fibrés vectoriels, dont la résolution permet, entre autres, de montrer l'existence d'analogues des lois de conservation en l'absence de symétries pour des équations aux dérivées partielles.

Les méthodes que nous utilisons pour tenter de répondre et de résoudre le problème du plongement isométrique généralisé sont géométriques : nous traduisons et exprimons le problème en termes de systèmes différentiels extérieurs, et la résolution du problème consistera à vérifier si le système différentiel extérieur est involutif, c'est-à-dire, montrer l'existence ou non de solutions, appelées dans ce langage, variétés intégrales. Les systèmes différentiels extérieurs constituent des outils importants en géométrie différentielle car ils permettent d'aborder par des moyens géométriques l'étude des équations aux dérivées partielles. En effet, tout système d'EDP peut s'exprimer en termes de système différentiel extérieur dans un certain espace de jets et réciproquement, et nous n'avons nul besoin de convaincre le lecteur de l'importance des EDP en mathématiques, et plus généralement en sciences.

L'appellation *problème du plongement isométrique généralisé* se justifie à la fois par des motivations que nous présentons dans ce qui suit, et par de nombreux faits surgissant lors de l'élaboration de la stratégie de résolution du problème dans le cas général, qui rappelle des phénomènes similaires observés à propos du *problème du plongement isométrique* (classique). Un des objets de la géométrie différentielle est l'étude de structures sur des variétés différentielles. Tout commence donc avec les travaux de Carl Friedrich Gauss [Gau27] qui a ouvert la voie dans l'étude des métriques de surfaces dans l'espace euclidien tri-dimensionnel. Nous lui devons, à la suite des travaux de Gaspard Monge, l'introduction de la première et seconde formes fondamentales, et un invariant intrinsèque : la courbure dite de Gauss (Theorema Egregium[3]). Plus tard, Georg Friedrich Bernhard Riemann révolutionne la géométrie : les notions abstraites de variété et de variété riemannienne en dimension quelconque voient le jour. Naturellement, la question s'est posée pour savoir s'il existe réellement des variétés abstraites ou si toute variété riemannienne n'est autre qu'une sous-variété d'un certain espace euclidien. Cette question, dont une réponse locale est fournie par le théorème de Cartan–Janet [Car27, Jan26] et dont la démonstration est présentée dans cette thèse (annexe du chapitre 3), peut s'exprimer d'une autre manière : est-il toujours possible de plonger isométriquement une variété riemannienne de dimension quelconque dans un espace euclidien?

[2]Les chapitres et annexes de cette présente thèse sont rédigés en anglais.

[3]Le théorème remarquable.

Nous rappelons que la résolution d'un cas particulier du problème du plongement isométrique généralisé, dont l'énoncé précis est présenté dans ce qui suit, est d'une certaine façon apparentée au problème de trouver des lois de conservation. En physique, une loi de conservation exprime qu'une quantité mesurable d'un système physique reste constante pendant l'évolution de ce système. De mémoire d'écolier, la célèbre loi de Lavoisier[4] « Rien ne se perd, rien ne se crée, tout se transforme » exprime la conservation de la matière. C'est aussi le cas pour de nombreuses propriétés fondamentales de la physique : l'énergie, la quantité de mouvement, le moment angulaire, la charge électrique, le flux magnétique, etc.

D'un point de vue mathématique et très général, une loi de conservation peut être vue comme une application définie sur un espace $\mathcal{F}$ (qui peut être, par exemple, un espace de fonctions, un espace de sections d'un fibré, au dessus d'une variété $\mathcal{M}$, etc.) qui associe à chaque élément f de $\mathcal{F}$, un champ de vecteurs tangents X sur une variété riemannienne $\mathcal{M}$ de dimension m, tel que si f est solution d'une équation aux dérivées partielles donnée, le champ de vecteurs associé est alors de divergence nulle. Si nous notons g la métrique riemannienne de la variété $\mathcal{M}$, nous pouvons canoniquement associer à chaque champ de vecteurs tangents $\mathrm{X} \in \Gamma(\mathrm{T}\mathcal{M})$, une 1-forme différentielle α_{X} définie par

$$\alpha_{\mathrm{X}} := g(\mathrm{X}, \cdot).$$

La divergence d'un champ de vecteurs tangents sur une variété riemannienne orientée est alors définie par

$$\mathrm{div}(\mathrm{X}) = *\mathrm{d} * \alpha_{\mathrm{X}} \qquad \text{ou bien par} \qquad \mathrm{div}(\mathrm{X})\mathrm{vol}_{\mathcal{M}} = \mathrm{d}(\mathrm{X} \lrcorner \, \mathrm{vol}_{\mathcal{M}}),$$

où $*$ est l'opérateur de Hodge, $\mathrm{vol}_{\mathcal{M}}$ est la m-forme volume sur $\mathcal{M}$, et $\mathrm{X} \lrcorner \, \mathrm{vol}_{\mathcal{M}}$ est la contraction de la m-forme $\mathrm{vol}_{\mathcal{M}}$ par le champ de vecteurs tangents X. Nous constatons que la condition $\mathrm{div}(\mathrm{X}) = 0$ peut être remplacée par $\mathrm{d}(\mathrm{X} \lrcorner \, \mathrm{vol}_{\mathcal{M}}) = 0$. Par conséquent, les lois de conservation peuvent être considérées comme des applications de $\mathcal{F}$ à valeurs dans les $(m-1)$-formes différentielles sur $\mathcal{M}$ telles que les solutions d'une EDP soient associées aux $(m-1)$-formes différentielles fermées de $\mathcal{M}$. Plus généralement, nous pouvons envisager d'étendre la notion de loi de conservation aux applications à valeurs dans les p-formes différentielles fermées (par exemple, les équations de Maxwell dans le vide peuvent s'exprimer par un système de 2-formes différentielles fermées).

Le théorème de Noether exprime l'équivalence qui existe entre les lois de conservation et l'invariance des lois physiques en ce qui concerne certaines transformations (typiquement appelées symétries). Nous nous intéressons au problème suivant, dont la formulation est due à Frédéric Hélein [Hél96], et dont le but est de trouver des lois de conservation pour une classe d'EDP décrite comme suit :

Problème 0.1 – Plongement isométrique généralisé Soit $\mathbb{V}$ un fibré vectoriel de rang n muni d'une métrique g, au dessus d'une variété différentielle de dimension m, et d'une connexion ∇ sur $\mathbb{V}$ respectant cette métrique. On note d_∇ la dérivée covariante induite par ∇ et agissant

[4]Il semble que Anaxagore de Clazomènes (500-428 av. J.C.) est à l'origine de cette citation et qu'elle est reprise par Antoine Lavoisier (1743-1749) dont la citation exacte est: « ... car rien ne se crée, ni dans les opérations de l'art, ni dans celles de la nature, et l'on peut poser en principe que, dans toute opération, il y a une égale quantité de matière avant et après l'opération ; que la qualité et la quantité des principes est la même, et qu'il n'y a que des changements, des modifications. »

sur les formes différentielles à valeurs dans $\mathbb{V}$. On suppose que ϕ est une p-forme différentielle sur $\mathcal{M}$, à valeurs dans $\mathbb{V}$, fermée covariante, i.e.,

$$\mathrm{d}_{\nabla}\phi = 0. \tag{1}$$

Trouver $\mathrm{N} \in \mathbb{N}$ et un plongement Ψ de $\mathbb{V}$ dans $\mathcal{M} \times \mathbb{R}^{\mathrm{N}}$ donné par $\Psi(x, \mathrm{X}) = (x, \Psi_x \mathrm{X})$, où Ψ_x est une application linéaire de $\mathbb{V}_x$ dans $\mathbb{R}^{\mathrm{N}}$ telle que:

- Ψ est isométrique, i.e, pour tout $x \in \mathcal{M}$, l'application $\Psi_x : \mathbb{V}_x \longrightarrow \mathbb{R}^{\mathrm{N}}$ est une isométrie,
- Si $\Psi(\phi)$ est l'image de ϕ par Ψ, i.e., $\Psi(\phi)_x = \Psi_x \circ \phi_x$ pour tout $x \in \mathcal{M}$, alors

$$\mathrm{d}\Psi(\phi) = 0. \tag{2}$$

L'équation (1) dans le problème du plongement isométrique généralisé représente le système d'EDP et la relation (2) joue le rôle de la loi de conservation. Dans le cas de fibrés vectoriels en droite réelle, le problème est trivial. En effet, la seule connexion compatible avec la métrique sur un fibré en droites réelles est la connexion plate.

MOTIVATIONS Il y a principalement deux motivations au problème du plongement isométrique généralisé de fibrés vectoriels:

LE PROBLÈME DU PLONGEMENT ISOMÉTRIQUE Le premier exemple fondamental est (tout naturellement) le problème du plongement isométrique des variétés riemanniennes dans l'espace euclidien, et est lié à notre problème comme suit : Lorsque $\mathcal{M}$ est une variété différentielle riemannienne de dimension m, le fibré vectoriel $\mathbb{V}$ est l'espace tangent $\mathrm{T}\mathcal{M}$, la connexion ∇ est la connexion de Levi-Civita, $p = 1$ et la 1-forme différentielle sur $\mathcal{M}$ à valeurs dans $\mathrm{T}\mathcal{M}$ fermée covariante ϕ est l'identité sur $\mathrm{T}\mathcal{M}$, alors (1) exprime le fait que la connexion ∇ est sans torsion. De plus, toute solution Ψ de (2) fournit un plongement isométrique u de la variété riemannienne $\mathcal{M}$ dans un espace euclidien $\mathbb{R}^{\mathrm{N}}$ par l'intégration de l'équation $\mathrm{d}u = \Psi(\phi)$ et réciproquement.

Le théorème de Cartan–Janet [Car27, Jan26], dont la preuve est exposée dans le chapitre 3, fournit, localement, une réponse positive au problème du plongement isométrique de variétés riemanniennes, dans le cas analytique. Il peut sembler que les hypothèses de localité et de régularité des données soient très restrictives, il n'en demeure pas moins que le résultat est important car la dimension de l'espace but est optimale, contrairement au plongement de Nash–Moser qui est un résultat global et dans le cas lisse.

Par conséquent, si le problème du plongement isométrique généralisé a une solution dans le cas $p = 1$, la notion de plongement isométrique est étendue à celle de plongement isométrique généralisé de fibrés vectoriels. Lorsque le degré de la forme différentielle à valeurs dans un fibré vectoriel fermée covariante est arbitraire, le plongement isométrique généralisé peut aussi être considéré comme un plongement de forme différentielle à valeurs dans un fibré vectoriel fermée covariante.

APPLICATIONS HARMONIQUES ENTRE VARIÉTÉS RIEMANNIENNES L'autre exemple fondamental traité dans [Hél96] de forme différentielle à valeurs dans un fibré vectoriel fermée covariante est fourni par les applications harmoniques entre deux variétés riemanniennes. Une application harmonique u entre deux variétés riemanniennes $(\mathcal{M}, g)$ et $(\mathcal{N}, h)$ est un point critique de la fonctionnelle de Dirichlet

$$\mathrm{E}[u] = \frac{1}{2}\int_{\mathcal{M}} |\mathrm{d}u|^2. \tag{3}$$

Localement, le système de Euler–Lagrange s'exprime comme suit[5]:

$$\Delta_g u^i + g^{\alpha\beta}\Gamma^i_{jk}(u(x))\frac{\partial u^j}{\partial x^\alpha}\frac{\partial u^k}{\partial x^\beta} = 0 \tag{4}$$

où Γ^i_{jk} désignent les symboles de Christoffel de la connexion sur $\mathcal{N}$. Le lecteur peut penser que les applications harmoniques ne sont pas si communes s'il n'a jamais rencontré cette définition. Néanmoins, les exemples abondent en mathématiques et en physique. Par exemple, lorsque la variété riemannienne but $(\mathcal{N}, h)$ est remplacée par $(\mathbb{R}, \langle,\rangle_{\mathbb{R}})$, les applications harmoniques sont les fonctions harmoniques de $(\mathcal{M}, g)$. Lorsque la variété but est $(\mathbb{R}^n, \langle,\rangle_{\mathbb{R}^n})$, une application u est harmonique si et seulement si chaque composante de u est une fonction harmonique de $\mathcal{M}$. D'autres exemples d'application harmoniques sont : les isométries, les géodésiques, les immersions isométriques et les applications holomorphes et anti-holomorphes entre variétés kähleriennes, dont certains sont traités dans le chapitre 4.

Les applications harmoniques fournissent des exemples de formes différentielles à valeurs dans des fibrés vectoriels fermées covariantes [Hél96]. En effet, les applications harmoniques entre variétés riemanniennes peuvent être caractérisées comme suit : soit u une application définie d'une variété riemannienne $(\mathcal{M}, g)$ de dimension m dans une variété riemannienne $(\mathcal{N}, h)$ de dimension n. Sur le fibré $u^*\mathrm{T}\mathcal{N}$ induit par u au dessus de $\mathcal{M}$, la $(m-1)$-forme différentielle $*\mathrm{d}u$ à valeurs dans $u^*\mathrm{T}\mathcal{N}$ est fermée covariante si et seulement si l'application u est harmonique, où la connexion sur le fibré induit est le pull-back par u de la connexion riemannienne sur $(\mathcal{N}, h)$. Par conséquent, si le problème du plongement isométrique généralisé a une solution dans le cas $p = m - 1$, il est alors possible de trouver l'analogue des lois de conservation sur $\mathcal{M}$ à partir de formes différentielles à valeurs dans des fibrés vectoriels fermées covariantes, en particulier, par celles produites par les applications harmoniques entre variétés riemanniennes.

Dans [Hél96], motivé par le problème de la compacité des applications faiblement harmoniques dans les espaces de Sobolev dans la topologie faible (qui semble toujours être un problème ouvert), Frédéric Hélein considère les applications harmoniques entre variétés riemanniennes, et explique comment obtenir explicitement des lois de conservation en utilisant le théorème de Noether dans le cas où les variétés buts sont symétriques, et formule le problème ci-dessus pour le cas de variétés riemanniennes buts non-symétriques.

LES RÉSULTATS Le premier résultat [Kah08b, Kah09], dont la preuve détaillée est présentée dans le chapitre 4 et 5, est une réponse positive dans le cas local et analytique au problème du plongement isométrique généralisé de fibrés vectoriels, lorsque $p = m - 1$. Comme pour le théorème de Cartan–Janet, nous donnons la dimension minimale de l'espace d'arrivée.

[5]Nous utilisons la convention de sommation d'Einstein.

Théorème 0.2 – Lois de conservation par plongement isométrique généralisé [Kah08b] Soit $\mathbb{V}$ un fibré vectoriel réel analytique de rang n au dessus d'une variété différentielle $\mathcal{M}$ réelle analytique de dimension m, muni d'une métrique analytique g et d'une connexion compatible avec g. Etant donnée ϕ une $(m-1)$-forme différentielle non nulle sur $\mathcal{M}$ à valeurs dans $\mathbb{V}$, fermée covariante, il existe un plongement isométrique local de $\mathbb{V}$ dans $\mathcal{M} \times \mathbb{R}^{n+\kappa^n_{m,m-1}}$ au dessus de $\mathcal{M}$ où $\kappa^n_{m,m-1} \geqslant (m-1)(n-1)$ de sorte que l'image de ϕ soit une loi de conservation.

Nous nous sommes rendus compte que ce résultat peut être appliqué pour trouver des lois de conservation à partir de 2-tenseurs contravariants à divergence covariante nulle, en particulier, pour le tenseur énergie-impulsion, qui joue un rôle important dans la relativité et la théorie de la gravitation. Pour ce faire, il faut voir le tenseur énergie-impulsion comme une $(m-1)$-forme différentielle à valeurs dans le fibré tangent, et vérifier qu'elle est fermée covariante. Il semble que cette manière de voir le tenseur énergie-impulsion n'est pas nouvelle, même si elle est en pratique peu utilisée, et apparaît dans les oeuvres d'Élie Cartan. En effet, soit $\mathrm{T} \in \Gamma(\mathrm{T}\mathcal{M} \otimes \mathrm{T}\mathcal{M})$ un 2-tenseur contravariant dont l'expression dans un repère mobile $(\xi_1, \ldots, \xi_m)$ est $\mathrm{T} = \mathrm{T}^{ij}\xi_i \otimes \xi_j$, où $(\xi_1, \ldots, \xi_m)$ est le duale d'un corepère $(\eta^1, \ldots, \eta^m)$. La m-forme de volume est notée par $\eta^{\mathrm{I}} = \eta^1 \wedge \cdots \wedge \eta^m$. En utilisant le produit intérieur, nous pouvons canoniquement associer à tout 2-tenseur contravariant T, une $(m-1)$-forme différentielle τ à valeurs dans $\mathrm{T}\mathcal{M}$ définie comme suit :

$$\begin{aligned}\Gamma(\mathrm{T}\mathcal{M} \otimes \mathrm{T}\mathcal{M}) &\longrightarrow \Gamma(\mathrm{T}\mathcal{M} \otimes \wedge^{(m-1)}\mathrm{T}^*\mathcal{M}).\\ \mathrm{T} = \mathrm{T}^{ij}\xi_i \otimes \xi_j &\longmapsto \tau = \xi_i \otimes \tau^i\end{aligned}$$

avec $\tau^i = \mathrm{T}^{ij}(\xi_j \lrcorner \eta^{\mathrm{I}})$. Nous montrons dans un lemme que τ est fermée covariante si et seulement si le tenseur T est de divergence covariante nulle, et par conséquent, le théorème 0.2 admet le corollaire suivant :

Corollaire 0.3 – Lois de conservation local du tenseur énergie-impulsion Soit $(\mathcal{M}^m, g)$ une variété différentielle réelle analytique de dimension m. Soit T un 2-tenseur contravariant de divergence covariante nulle. Il existe alors une loi de conservation de T dans $\mathcal{M} \times \mathbb{R}^{m+(m-1)^2}$.

Le problème du plongement isométrique généralisé reste ouvert lorsque le degré p de la forme différentielle à valeurs dans un fibré vectoriel est strictement inférieur à $(m-1)$. Néanmoins, nous avons obtenu des résultats partiels pour les cas $p = 1$ et $p = 2$.

Théorème 0.4 – Le cas $(\mathbb{V}^2, \mathcal{M}^m, g, \nabla, \phi)_1$. Soit $\mathbb{V}^2$ un fibré vectoriel réel analytique de rang 2 au dessus d'une variété différentielle réelle analytique $\mathcal{M}$ de dimension m, muni d'une métrique g et d'une connexion ∇ compatible avec g. Étant donnée une 1-forme différentielle ϕ à valeurs dans $\mathbb{V}$ fermée covariante non-nulle et non-dégénérée, il existe un plongement isométrique généralisé local de $\mathbb{V}^2$ dans $\mathcal{M} \times \mathbb{R}^{n+\kappa^2_{m,1}}$ au dessus $\mathcal{M}$, où $\kappa^n_{m,m-1} \geqslant 1$, tel que l'image de ϕ soit une loi de conservation.

Nous expliquons dans le chapitre 6 comment prouver le résultat dans le cas d'un fibré de rang quelconque, i.e., n et m arbitraires et $p = 1$. Notons aussi qu'une preuve est présentée dans [Hél09] dans le cas où la 1-forme différentielle à valeurs dans un fibré vectoriel fermée covariante est bijective, injective ou surjective, ou plus généralement de rang constant. De plus, Frédéric Hélein utilise les ingrédients du plongement isométrique généralisé, à savoir le fibré vectoriel $\mathbb{V}^n$ de rang n, la variété différentielle $\mathcal{M}^m$ de dimension m, la métrique g, la connexion ∇ (compatible avec g) et la p-forme différentielle ϕ à valeurs dans $\mathbb{V}$ fermée covariante, que nous notons par le 5-uplet $(\mathbb{V}^n, \mathcal{M}^m, g, \nabla, \phi)_p$, pour définir des géométries par recollement de points, lignes, surfaces etc. La forme différentielle ϕ joue ainsi le rôle d'une forme de soudure et les ingrédients $(\mathbb{V}^n, \mathcal{M}^m, g, \nabla, \phi)_p$ sont associés aux $(p-1)$-puzzles.

Le théorème suivant est une réponse positive au problème du plongement isométrique généralisé dans le cas $p = 2$ pour un fibré vectoriel de rang 3 au dessus d'une variété différentielle réelle analytique de dimension 4, qui est une dimension très importante en physique, muni d'une connexion anti-auto-duale. Ce cas est lié au 1-puzzle de ces géométries "upstairs" décrites dans [Hél09].

Théorème 0.5 – Plongement isométrique généralisé de 2-formes avec condition d'anti-auto-dualité Soit $\mathcal{M}^4$ une variété réelle analytique de dimension 4. Soit $\mathbb{V}^3$ un fibré vectoriel réel analytique de rang 3 au dessus de $\mathcal{M}^4$, muni d'une métrique riemannienne g, d'une connexion anti-auto-dual g-compatible ∇, et d'une 2-forme différentielle à valeurs dans $\mathbb{V}^3$ covariante fermée ϕ de la forme (6.3). Il existe alors un plongement isométrique généralisé Ψ de $\mathbb{V}^3$ dans $\mathcal{M}^4 \times \mathbb{R}^{3+\kappa^3_{4,2,\mathrm{ASD}}}$, où $\kappa^3_{4,2,\mathrm{ASD}} \geqslant 4$, de sorte que $\Psi(\phi)$ soit une loi de conservation locale.

STRATÉGIE DE RÉSOLUTION En utilisant le formalisme de Cartan, nous traduisons le problème du plongement isométrique en termes de formes différentielles, et nous montrons ainsi qu'il est équivalent à résoudre un système différentiel extérieur sur une variété définie à partir des données du problème. Pour montrer l'existence de variétés intégrales, il faut vérifier que le système différentiel extérieur est fermé par rapport à la différentiation extérieure. Il s'avère que ce n'est pas le cas. Nous devons ajouter donc les différentielles extérieures de toutes les formes différentielles qui engendrent l'idéal extérieur pour ainsi obtenir un système différentiel fermé.

La grande difficulté dans ce problème est le fait que plusieurs objets auxquels nous sommes confrontés ont un sens géométrique dans le cas du fibré tangent et de la 1-forme différentielle standard ($\phi = \mathrm{Id}_{T\mathcal{M}}$), mais pas sur un fibré vectoriel quelconque. Ce constat nous mène à définir ces objets et notions dans un sens généralisé de sorte qu'ils coïncident avec les notions standards dans le cas du fibré tangent. Nous définissons tout d'abord la 2-forme de torsion généralisée qui mène aux identités de Bianchi généralisées, et qui caractérisent en un sens les géométries liées aux p-puzzles. Le lemme de Cartan, qui dans le problème du plongement isométrique (classique) exprime la symétrie des coefficients de la seconde forme fondamentale, ne s'applique pas lorsque le degré de la forme différentielle fermée covariante est différent de 1. Les relations entre ces coefficients sont données par les identités de Cartan généralisées. Finalement, nous définissons l'analogue des équations de Gauss.

Le coeur de la démonstration du théorème 4.12 est le lemme fondamentale 5.14, et ce pour deux raisons : D'une part, il assure l'existence de coefficients de la seconde forme fondamentale qui satisfont les identités de Cartan généralisées et les équations de Gauss généralisées, et d'autre part, le lemme fournit la codimension $\kappa^n_{m,m-1}$ qui assure l'existence du plongement isométrique généralisé. Une autre démonstration du théorème 4.12 est donnée dans le chapitre 5 par une construction explicite d'un drapeau intégral. Enfin, lorsque l'existence de variétés intégrales est établie, il ne reste qu'à les projeter sur $\mathcal{M} \times \mathbb{R}^{n+\kappa^n_{m,p}}$.

DESCRIPTIF DES CHAPITRES ET ANNEXES Cette présente thèse se compose de six chapitres et de deux annexes dont le contenu est:

CHAPITRE 1 Le but de ce chapitre est d'établir les équations de structure de Cartan. Les notions de base en géométrie différentielle sont introduites dans le langage de Cartan, c'est-à-dire en utilisant les formes différentielles et les repères mobiles. Nous introduisons dans la première section aux notions de 1-forme de connexion et de la 2-forme de courbure sur un fibré vectoriel quelconque au dessus d'une variété différentielle. La relation entre ces deux objets donne lieu à la seconde équation de structure de Cartan et aux identités de Bianchi. Nous montrons aussi une propriété intéressante des formes de connexion et de courbure lorsque le fibré vectoriel est munit d'une métrique, et comme toujours en géométrie différentielle, les règles de transformations par changement de coordonnées sont bien entendu spécifiées pour tous les objets définis. Dans la seconde section, nous nous spécialisons au cas des fibrés tangents de variétés différentielles. Nous définissons la 2-forme de torsion d'une connexion et nous établissons la première équation de structure de Cartan ainsi que la relation entre la 1-forme de connexion, la 2-forme de courbure, la 2-forme de torsion et le corepère mobile. Les démonstrations et résultats des deux premières sections sont donnés en annexe 1. La section 3 est dédiée à l'exploitation des équations de structure de Cartan et des formes différentielles dans l'étude des surfaces : à partir des métriques riemanniennes, nous donnons l'expression de la 1-forme de connexion, des symboles de Christoffel et de la courbure de Gauss. Enfin, nous présentons un problème étudié par Henri Poincaré concernant l'existence de métriques conformes à courbure de Gauss constantes.

Chapitre 2 Nous introduisons dans ce chapitre les systèmes différentiels extérieurs qui sont une manière géométrique de voir les systèmes d'équations aux dérivées partielles, et la théorie de Cartan–Kähler, qui est l'outil utilisé pour montrer l'existence ou non de solutions. La première section est dédiée donc à l'introduction aux systèmes différentiels extérieurs, aux idéaux différentiels extérieurs, aux variétés intégrales d'un système différentiel extérieur et à la notion d'involution. Nous énonçons le théorème de Frobenius via les formes différentielles qui fournit une condition nécessaire et suffisante d'involution des systèmes de Pfaff. La seconde section est dédiée à la théorie de Cartan–Kähler, qui permet de montrer l'existence ou non des variétés intégrales d'un idéal extérieur. Nous commençons par définir les éléments intégraux d'un idéal extérieur, leurs espaces polaires et leurs rangs d'extension. Ensuite, nous nous intéressons aux systèmes différentiels extérieurs ayant une condition d'indépendance. Nous énonçons le test d'involution de Cartan et une proposition qui permet de calculer les caractères nécessaires au test de Cartan. Enfin, nous énonçons deux théorèmes d'existence : le théorème de Cauchy–Kowalevskaya pour l'existence de solutions de systèmes différentiels et une généralisation de ce résultat, le théorème de Cartan–Kähler pour l'existence des variétés intégrales d'un idéal différentiel extérieur.

Chapitre 3 Nous présentons dans ce chapitre différents résultats de plongement de surfaces riemanniennes : le plongement lagrangien, le plongement isométrique et le plongement isométrique lagrangien de surfaces riemanniennes. Le but est à la fois de présenter des applications géométriques importantes des deux premiers chapitres, en particulier la théorie de Cartan–Kähler, et de familiariser le lecteur avec les techniques que nous utiliserons dans les prochains chapitres, qui sont destinées à la compréhension et à la résolution de quelques cas du problème du plongement isométrique généralisé de fibrés vectoriels. Ce chapitre comporte une annexe où nous exposons et démontrons les résultats de plongements qui se généralisent en dimensions supérieures, à savoir l'existence de variétés lagrangiennes et le théorème de Cartan–Janet.

Chapitre 4 Nous posons dans ce chapitre le problème du plongement isométrique généralisé de fibrés vectoriels et nous montrons ses liens, d'une part avec le problème du plongement isométrique de variétés riemanniennes, et d'autre part, avec les lois de conservation et les applications harmoniques entre variétés riemanniennes. La section 1 est donc dédiée à définir mathématiquement les lois de conservation via les champs de vecteurs puis via les formes différentielles. Nous énonçons dans la section 2 le problème du plongement isométrique généralisé de fibrés vectoriels. Dans la section 3, nous présentons en détails les motivations principales du problème. Tout d'abord, le lien entre le problème classique et le problème généralisé est exposé. Ensuite, nous définissons les applications harmoniques entre deux variétés riemanniennes, nous donnons plusieurs exemples, et nous exposons le lien avec notre problème. Dans la section 4 sont regroupés les résultats de cette thèse concernant le problème du plongement isométrique généralisé. Finalement, la section 5 est consacrée à une application aux tenseurs énergie-impulsion. Une annexe de chapitre est consacrée à des détails supplémentaires sur la démonstration d'un lemme utilisé dans la preuve du corollaire des lois de conservation pour les tenseurs énergie-impulsion dans le cas de surfaces.

Chapitre 5 Nous détaillons dans ce chapitre la stratégie de résolution du problème du plongement isométrique généralisé dans le cas général et nous traitons le cas des lois de conservation, c'est-à-dire lorsque $p = m - 1$. Ce chapitre est basé sur [Kah08b, Kah09]. Dans la section 1, nous traduisons le problème du plongement isométrique généralisé en termes de système différentiel extérieur. Nous définissons les notions de torsion généralisée, les identités de Bianchi généralisées, l'espace des tenseurs de courbure généralisé, les identités de Cartan généralisées, et l'application de Gauss généralisée. La section 2 est consacrée à la démonstration du théorème 0.2 dont le point clef réside dans le lemme 5.14. Par ailleurs, une autre preuve est présentée par construction explicite d'un drapeau intégral ordinaire. La démonstration (très technique) du lemme 5.14 est exposée pour plusieurs sous-cas afin d'aider le lecteur à comprendre les détails techniques.

Chapitre 6 Finalement, nous présentons les autres résultats du problème du plongement isométrique généralisé: le cas d'une 1-forme différentielle à valeurs dans un fibré vectoriel, et le cas d'une 2-forme sur une variété de dimension 4 à valeurs dans un fibré de rang 3 et d'une connection anti-auto-duale. L'annexe de chapitre est dédiée à présenter une autre expression des identités de Bianchi généralisé pour le cas $n = 2, m = 3$ et $p = 1$ ainsi que pour $n = 3, m = 3$ et $p = 1$, et ce en définissant un produit vectoriel à valeurs vectorielles.

Annexe 1 Nous avons regroupé dans la première annexe les démonstrations et les calculs techniques des résultats énoncés dans la première et seconde section du chapitre 1. Bien que ces résultats soient classiques, nous avons tenu à les inclure ici à la fois pour ne pas alourdir le chapitre mais aussi dans le but d'être complet.

Annexe 2 Nous introduisons brièvement dans cette seconde et dernière annexe à la théorie de systèmes de Pfaff linéaires et à la théorie de Cartan–Kähler via les tableaux. Nous présentons quelques exemples d'application : l'équation de la chaleur sur $\mathbb{R}^2$, le plongement conforme et l'existence des variétés lagrangiennes dans les espaces complexes $\mathbb{C}^m$.

Perspectives En ce qui concerne le théorème 0.2, il est assez naturel de se demander, comme pour le problème du plongement isométrique de variétés riemanniennes, si les hypothèses de régularité peuvent être assouplies, i.e., avoir le même théorème avec des données $\mathcal{C}^\infty$ ou $\mathcal{C}^k$. De plus, nous pensons qu'il est possible d'avoir un plongement isométrique généralisé global en utilisant le principe d'homotopie de Gromov. Bien entendu, nous nous attendons dans les deux cas que la dimension de l'espace d'arrivée soit plus grande.

Le problème reste ouvert dans plusieurs cas, $p = 2, \ldots, m-2$. Néanmoins se pose la question de l'existence d'obstructions dans certains cas et s'il est possible de construire des contre-exemples explicites. Enfin, qu'en est-t-il du cas de la forme fermée covariante de rang non constant?

Introduction (in english)[6]

This thesis pertains to the field of differential geometry. Its main objective is the study of the generalized isometric embedding problem of vector bundles, whose solutions lead, among other things, to show the existence of the analogeous of conservation laws when there are no symmetries for partial differential equation.

The methods used to answer and solve the generalized isometric embedding problem are geometric: the problem is translated and expressed in terms of an exterior differential system, and solving the problem then consists of investigating if the exterior differential system is involutive, i.e., showing the existence (or not) of solutions, called in this language, integral manifolds. Exterior differential systems are important tools in differential geometry because they allow the study of PDEs geometrically. Indeed, any system of PDEs can be expressed in terms of an exterior differential system on a specific jet space, and conversely, and there is no need to convince the reader of the importance of PDEs in mathematics, and more generally, in science.

The denomination *generalized isometric embedding problem* is justified not only by the motivations (presented below), but also by numerous facts arising from the elaboration of a strategy to solve the generalized isometric embedding problem in the general case, which calls to mind similar observed phenomenon of the (classical) isometric embedding problem. One of the goals of differential geometry is the study of structures on differential manifolds. All began with the work of Carl Friedrich Gauss [Gau27], who opened the way for the study of metrics of surfaces in the 3-dimensional Euclidean space. We owe him, after the work of Gaspard Monge, the introduction of the first and second fundamental forms, and an intrinsic invariant: the so-called Gauss curvature (Theorema Egregium[7]). Later, Georg Friedrich Bernhard Riemann revolutionized the field of geometry: the notions of manifolds and Riemannian manifolds of higher dimensions appear. Naturally, the question arose of the existence of abstract Riemannian manifolds. In other words, is any abstract Riemannian manifold nothing more than a submanifold of a given Euclidean space? This question, whose local answer is provided by the Cartan–Janet theorem [Car27, Jan26] among others, and whose proof is presented in this present thesis (appendix of chapter 3), may be expressed in another way: is it always possible to isometrically embed any Riemannian manifold in a Euclidean space?

Let us recall that solving a particular case in the generalized isometric embedding problem, for which a precise statement is presented in what follows, is somehow related to the problem

[6] Chapters and appendices of the present thesis are written in English.
[7] The remarkable theorem.

of finding conservation laws. In physics, a conservation law expresses that a given measurable quantity of a physical system remains constant during the evolution of the system. This is the case for numerous fundamental quantities in physics, such as energy, movement quantity, momenta, electric charge, magnetic fields, etc.

From a mathematical and a very general viewpoint, a conservation law can be seen as a map defined on a space $\mathcal{F}$ (which can be, for instance, a function space, a cross section space, etc.) that associates each element f of $\mathcal{F}$ with a tangent vector field X on a Riemannian manifold $\mathcal{M}$ of dimension m, such that if f is a solution to a given PDE, then the tangent vector field has a vanishing divergence. If the metric of the Riemannian manifold $\mathcal{M}$ is denoted by g, then we can canonically associate every tangent vector field $\mathrm{X} \in \Gamma(\mathrm{T}\mathcal{M})$ with a differential 1-form α_{X} defined by

$$\alpha_{\mathrm{X}} := g(\mathrm{X}, \cdot).$$

The divergence of a tangent vector field on an oriented Riemannian manifold is then defined either by

$$\mathrm{div}(\mathrm{X}) = *\mathrm{d} * \alpha_{\mathrm{X}} \qquad \text{or} \qquad \mathrm{div}(\mathrm{X})\mathrm{vol}_{\mathcal{M}} = \mathrm{d}(\mathrm{X} \lrcorner \, \mathrm{vol}_{\mathcal{M}}),$$

where $*$ is the Hodge operator, $\mathrm{vol}_{\mathcal{M}}$ is the volume differential m-form on $\mathcal{M}$, and $\mathrm{X} \lrcorner \, \mathrm{vol}_{\mathcal{M}}$ is the contraction of the m-form $\mathrm{vol}_{\mathcal{M}}$ by the tangent vector field X. One can notice that the requirement $\mathrm{div}(\mathrm{X}) = 0$ may be replaced by $\mathrm{d}(\mathrm{X} \lrcorner \, \mathrm{vol}_{\mathcal{M}}) = 0$. Therefore, a conservation law may been seen as a mapping from $\mathcal{F}$ to the differential $(m-1)$-forms on $\mathcal{M}$ such that the solutions to a PDE are associated with the closed differential $(m-1)$-forms on $\mathcal{M}$. More generally, one can foresee extending the notion of conservation law as a mapping to the closed differential p-forms (for instance, the Maxwell equation in vacuum can be expressed by requiring a system of differential 2-forms to be closed).

The Noether theorem expresses the equivalence that exists between conservation laws and the invariance of the physical laws for some transformations (called symmetries). This thesis is interested in the problem stated by Frédéric Hélein [Hél96] of finding conservation laws for a class of PDEs expressed as follows:

Problem 0.6 – The generalized isometric embedding problem Let $\mathbb{V}$ be an n-dimensional vector bundle over $\mathcal{M}$. Let g be a metric on the bundle and ∇ a connection that is compatible with that metric. We then have a covariant derivative d_∇ acting on vector bundle valued differential forms. Assume that ϕ is a given covariantly closed $\mathbb{V}$-valued differential p-form on $\mathcal{M}$, i.e.,

$$\mathrm{d}_\nabla \phi = 0. \tag{5}$$

Does there exist $\mathrm{N} \in \mathbb{N}$ and an embedding Ψ of $\mathbb{V}$ into $\mathcal{M} \times \mathbb{R}^{\mathrm{N}}$ given by $\Psi(x, \mathrm{X}) = (x, \Psi_x \mathrm{X})$, where Ψ_x is a linear map from $\mathbb{V}_x$ to $\mathbb{R}^{\mathrm{N}}$ such that:

- Ψ is isometric, i.e, for every $x \in \mathcal{M}$, the map $\Psi_x : \mathbb{V}_x \longrightarrow \mathbb{R}^{\mathrm{N}}$ is an isometry,
- If $\Psi(\phi)$ is the image of ϕ by Ψ, i.e., $\Psi(\phi)_x = \Psi_x \circ \phi_x$ for all $x \in \mathcal{M}$, then

$$\mathrm{d}\Psi(\phi) = 0. \tag{6}$$

The equation (5) in the generalized isometric embedding problem represents the system of PDEs, and the relation (6) plays the role of the conservation law. Note that the generalized isometric embedding problem is trivial for real line bundles. Indeed, the only connection compatible with the metric in a line bundle is the flat one.

MOTIVATIONS There are basically two main motivations for the generalized isometric embedding problem:

THE ISOMETRIC EMBEDDING PROBLEM The first fundamental example is (naturally) the isometric embedding problem of Riemannian manifolds in Euclidean spaces, and is related to our problem as follows: When $\mathcal{M}$ is a differential manifold of dimension m, the vector bundle $\mathbb{V}$ is the tangent space $\mathrm{T}\mathcal{M}$, the connection ∇ is the Riemannian connection, $p = 1$ and the covariantly closed differential 1-form with value on $\mathrm{T}\mathcal{M}$ is the identity map on $\mathrm{T}\mathcal{M}$, then equation (5) expresses the torsion-free condition of the connection ∇. Moreover, every solution Ψ to (6) provides us with an isometric embedding u of the Riemannian manifold $\mathcal{M}$ in an Euclidean space $\mathbb{R}^{\mathrm{N}}$ through the integration of the equation $\mathrm{d}u = \Psi(\phi)$, and conversely.

The Cartan–Janet theorem [Car27, Jan26], whose proof is given in chapter 3, provides locally a positive answer to the isometric embedding problem of Riemannian manifolds in the analytic category. Although the regularity condition on the data may seem to be restrictive, the result is fundamental because the dimension of the target Euclidean space is optimal. This result contrasts with the Nash–Moser result, which is a global result in the smooth category.

Consequently, if the generalized isometric embedding problem has a solution when $p = 1$, the notion of isometric embedding is extended to the generalized isometric embedding of vector bundles. When the degree of the covariantly closed vector bundle valued differential form is arbitrary, the generalized isometric embedding may also be considered to be an embedding of the vector bundle valued differential form.

HARMONIC MAPS BETWEEN RIEMANNIAN MANIFOLDS Another fundamental example of covariantly closed vector bundle valued differential forms (expounded in [Hél96]) is provided by harmonic maps between Riemannian manifolds. A harmonic map u between two Riemannian manifolds $(\mathcal{M}, g)$ and $(\mathcal{N}, h)$ is a critical point of the Dirichlet functional

$$\mathrm{E}[u] = \frac{1}{2}\int_{\mathcal{M}} |\mathrm{d}u|^2. \tag{7}$$

Locally, the Euler–Lagrange system is expressed as follows[8]:

$$\Delta_g u^i + g^{\alpha\beta}\Gamma^i_{jk}(u(x))\frac{\partial u^j}{\partial x^\alpha}\frac{\partial u^k}{\partial x^\beta} = 0 \tag{8}$$

where Γ^i_{jk} are the Christoffel symbols of the connection on $\mathcal{N}$. When reading the definition for the first time, the reader may think that harmonic maps are not common. Nevertheless many examples abound not only in mathematics but also in physics. For instance, when the target Riemannian manifold $(\mathcal{N}, h)$ is $(\mathbb{R}, \langle,\rangle_{\mathbb{R}})$, harmonic maps are harmonic functions on $(\mathcal{M}, g)$. When the target Riemannian manifold is $(\mathbb{R}^n, \langle,\rangle_{\mathbb{R}^n})$, then a map u is harmonic if and only if each component of u is a harmonic function of $\mathcal{M}$. Other examples of harmonic maps are: isometries, geodesic parameterization, isometric immersion, and holomorphic and

[8]We use the Einstein summation convention.

anti-holomorphic maps between Kählerian manifolds. Some of these examples are expounded in chapter 4.

Harmonic maps also provides examples of covariantly closed vector bundle valued differential forms [Hél96]. Indeed, harmonic maps between Riemannian manifolds can be characterized in the following way: Let u be a map defined on an m-dimensional Riemannian manifold $(\mathcal{M}, g)$ with values in a n-dimensional Riemannian manifold $(\mathcal{N}, h)$. On the vector bundle over $\mathcal{M}$ induced by u , the $u^*\mathrm{T}\mathcal{N}$-valued differential $(m-1)$-form $*\mathrm{d}u$ is covariantly closed if and only if the map u is harmonic, where the connection on the induced bundle $u^*\mathrm{T}\mathcal{N}$ is the pull-back by u of the Riemannian connection of $(\mathcal{N}, h)$. Therefore, if the generalized isometric embedding problem has a solution when $p = m-1$, it would then be possible to find the analogous of conservation laws on $\mathcal{M}$ from covariantly closed vector bundle valued differential forms, and in particular, those provided by harmonic maps between Riemannian manifolds.

In [Hél96], motivated by the problem of the compactness of weakly harmonic maps in Soblev spaces in the weak topology (which remains an open problem), Frédéric Hélein considers harmonic maps between Riemannian manifolds, explains how conservation laws may be explicitly obtained by using the Noether theorem when the target Riemannian manifold is symmetric, and formulates the above problem for non-symmetric Riemannian manifolds.

RESULTS The first result [Kah08b, Kah09], for which a detailed proof is later presented in chapters 4 and 5, is a positive local and analytic answer to the generalized isometric embedding problem when $p = m-1$ (also called the conservation law case). As for the Cartan–Janet theorem, we provide the minimal dimension of the embedding target space.

Theorem 0.7 – Local conservation laws by generalized isometric embeddings [Kah08b] Let $\mathbb{V}$ be a real analytic n-dimensional vector bundle over a real analytic m-dimensional manifold $\mathcal{M}$ endowed with a metric g and a connection ∇ compatible with g. Given a non-vanishing covariantly closed $\mathbb{V}$-valued differential $(m-1)$-form ϕ, there exists a local isometric embedding of $\mathbb{V}$ in $\mathcal{M} \times \mathbb{R}^{n+\kappa^n_{m,m-1}}$ over $\mathcal{M}$, where $\kappa^n_{m,m-1} \geqslant (m-1)(n-1)$ such that the image of ϕ is a conservation law.

We noticed that this result can also be applied to finding conservation laws for contravariant 2-tensors with a vanishing covariant divergence, and in particular, for the energy-momentum tensor which plays an important role in the general relativity and gravitation theory. For that purpose, we see the energy-momentum tensor as a differential $(m-1)$-form with values in the vector bundle, and we check that the form is covariantly closed. It seems that this way of looking at the energy-momentum tensor is not new, although it is in rarely used in practice, and appears in Élie Cartan's works. Indeed, let $\mathrm{T} \in \Gamma(\mathrm{T}\mathcal{M} \otimes \mathrm{T}\mathcal{M})$ be a contravariant 2-tensor that is expressed in a coordinate system by $\mathrm{T} = \mathrm{T}^{ij}\xi_i \otimes \xi_j$, where $(\xi_1, \ldots, \xi_m)$ is a moving frame dual to the moving coframe $(\eta^1, \ldots, \eta^m)$. The volume differential m-form is denoted by $\eta^{\mathrm{I}} = \eta^1 \wedge \cdots \wedge \eta^m$. Using the interior product, we can canonically associate any contravariant 2-tensor T with a $\mathrm{T}\mathcal{M}$-valued differential $(m-1)$-form τ as follows:

$$\begin{aligned} \Gamma(\mathrm{T}\mathcal{M} \otimes \mathrm{T}\mathcal{M}) &\longrightarrow \Gamma(\mathrm{T}\mathcal{M} \otimes \wedge^{(m-1)}\mathrm{T}^*\mathcal{M}). \\ \mathrm{T} = \mathrm{T}^{ij}\xi_i \otimes \xi_j &\longmapsto \tau = \xi_i \otimes \tau^i \end{aligned}$$

where $\tau^i = \mathrm{T}^{ij}(\xi_j \lrcorner \eta^{\mathrm{I}})$. We prove in a lemma that τ is covariantly closed if and only if the tensor T has a vanishing covariant divergence, and hence, Theorem 0.2 leads to the following corollary:

Corollary 0.8 – Local conservation laws for divergence-free contravariant 2-tensors Let $(\mathcal{M}^m, g)$ be a real analytic m-dimensional Riemannian manifold, ∇ be the Levi-Civita connection and T be a contravariant 2-tensor with a vanishing covariant divergence. Then there exists a conservation law for T on $\mathcal{M} \times \mathbb{R}^{m+(m-1)^2}$.

The generalized isometric embedding problem remains open when the degree of the covariantly closed vector bundle valued differential form is less than $(m-1)$. However, we obtained partial results when $p = 1$ and $p = 2$.

Theorem 0.9 – $(\mathbb{V}^2, \mathcal{M}^m, g, \nabla, \phi)_1$ case Let $\mathbb{V}^2$ be a real analytic 2-dimensional vector bundle over a real analytic m-dimensional manifold $\mathcal{M}$ endowed with a metric g and a connection ∇ compatible with g. Given a non-vanishing covariantly closed non-degenerate $\mathbb{V}$-valued differential 1-form ϕ, there exists a local isometric embedding of $\mathbb{V}^2$ in $\mathcal{M} \times \mathbb{R}^{n+\kappa^2_{m,1}}$ over $\mathcal{M}$, where $\kappa^n_{m,m-1} \geqslant 1$ such that the image of ϕ is a conservation law.

We explain in chapter 6 how to prove the above result for a vector bundle of an arbitrary rank , i.e., n and m arbitrary and $p = 1$. Let us notice that a proof is also presented in [Hél09] when the covariantly closed differential 1-form is bijective, injective, surjective, and, more generally, of constant rank. Moreover, Frédéric Hélein uses the ingredients of the generalized isometric embedding problem, namely the vector bundle $\mathbb{V}^n$ of rank n, the differential manifold $\mathcal{M}^m$ of dimension m, the metric g, the connection ∇ (compatible with g) and the covariantly closed $\mathbb{V}^n$-valued differential p-form ϕ, that we denote by the 5-uplet $(\mathbb{V}^n, \mathcal{M}^m, g, \nabla, \phi)_p$, to define geometries by gluing points, lines, surfaces, etc. The differential form ϕ plays the role of a solder form and the ingredients $(\mathbb{V}^n, \mathcal{M}^m, g, \nabla, \phi)_p$ are associated with $(p-1)$-puzzles.

The following theorem is a positive answer to the generalized isometric embedding problem in the case of a vector bundle of rank 3 over a real analytic differential manifold of dimension 4 (which is an important dimension in physics), $p = 2$, and an anti-self-dual connection. This case is related to the 1-puzzle of the "upstairs" geometries described in [Hél09].

Theorem 0.10 – Generalized isometric embedding of 2-form with anti-self dual condition Let $\mathcal{M}^4$ be an oriented real analytic 4-dimensional manifold endowed with a metric (actually a conformal structure is enough). Consider a real analytic vector bundle $\mathbb{V}^3$ of rank 3 over $\mathcal{M}^4$, endowed with a Riemannian metric g, an anti-self-dual g-compatible connection ∇, and a covariantly closed $\mathbb{V}^3$-valued differential 2-form ϕ of the form (6.3). There exists then a generalized isometric embedding Ψ of $\mathbb{V}^3$ into $\mathcal{M}^4 \times \mathbb{R}^{3+\kappa^3_{4,2,\mathrm{ASD}}}$, where $\kappa^3_{4,2,\mathrm{ASD}} \geqslant 4$, such that $\Psi(\phi)$ is a local conservation law.

Strategy of solving Using Cartan's formalism, the generalized isometric embedding problem is translated in terms of differential forms, and we show that this problem is equivalent to solving an exterior differential system on a manifold constructed from the problem's data. To show the existence of integral manifolds, we have to check that the exterior differential system is closed under the exterior differentiation. However, this is not the case, and we obtain a closed exterior differential system by adding the exterior differential of all of the differential forms.

The big difficulty of the generalized isometric embedding problem is due to the fact that various object and notions with which we are dealing have a geometric meaning in the tangent bundle case and with the standard differential 1-form ($\phi = \mathrm{Id}_{\mathrm{T}\mathcal{M}}$), but not on a general vector bundle. This situation leads us to define these objects and notions in a generalized sense in such a way that we recover the standard objects and notions in the tangent bundle. Thus, we first define the generalized torsion that leads to the generalized Bianchi identities that characterize the "upstairs" geometries and the p-puzzles. The Cartan lemma, which expresses in the (classic) isometric embedding problem the symmetry of the coefficient of the second fundamental form, does not hold when the degree of the covariantly closed vector bundle valued differential form is different than 1. The relations between these coefficients are given by the generalized Cartan identities. Finally, we define the analogous of the Gauss equation.

The key to Theorem 4.12's proof is the fundamental lemma 5.14 for two main reasons: On one hand, it assures the existence of suitable coefficients of the second fundamental form that satisfy both generalized Cartan identities and the generalized Gauss equations. On the other hand, it provides the minimum embedding codimension $\kappa^n_{m,m-1}$ that assures the existence of a generalized isometric embedding. An additional proof of Theorem 4.12 is given in chapter 5 by an explicit construction of an ordinary integral flag. Finally, when the existence of integral manifolds is established, we then need to merely project them on $\mathcal{M} \times \mathbb{R}^{n+\kappa^n_{m,p}}$.

Chapters and appendices description The present thesis is composed of six chapters and two appendices:

Chapter 1 The goal of this chapter is to establish Cartan's structure equations. The main notions of differential geometry are introduced in Cartan's language, i.e., in terms of differential forms and moving coframes. In section 1, the connection 1-form and the curvature 2-form of a connection are introduced on an arbitrary vector bundle above a differential manifold. The relation between these objects is expressed by Cartan's second-structure equation and the Bianchi identity. Also shown is an interesting property of the connection and curvature forms when the vector bundle is endowed with a Riemannian metric, and as always in differential geometry, the transformation rules are established for all of the defined objects. In section 2, we specialize in the tangent bundle case. The torsion 2-form of the connection is defined, Cartan's first-structure equation is established, and we give the relation between the connection 1-form, the curvature 2-form, the torsion 2-form, and the moving coframe. The proofs of the results in sections 1 and 2 are given in appendix 1. Section 3 is dedicated to making the most of Cartan's structure equations and differential forms for the study of surfaces: from Riemannian metrics, we give the expression of the connection 1-form, the Christoffel symbols and the Gauss curvature. Finally, we present a problem studied by Henri Poincaré pertaining to the existence of conformal metrics with constant Gauss curvature.

Chapter 2 In this chapter, we introduce exterior differential systems, which are merely a geometric way of studying PDEs, and the Cartan–Kähler theory, which is a powerful tool to show the existence (or not) of solutions. Section 1 is then dedicated to introducing exterior differential systems, exterior differential ideals, integral manifolds of an exterior differential system, and the notion of involution. We state the Frobenius theorem via differential forms, which provides us with a necessary and sufficient condition for the involution of a Pfaffian system. Section 2 is dedicated to Cartan–Kähler theory, which allows us to show the existence (or not) of integral manifolds of an exterior differential ideal. Defined are also integral elements, their polar spaces, and their extension ranks. Then we consider exterior differential systems that possess an independence condition. For the involution, the Cartan test is stated as well as a proposition which allows us to compute the Cartan characters in order to apply Cartan's test. For the existence of integral manifolds, we state Cauchy–Kowalevskaya theorem and the Cartan–Kähler theorem.

Chapter 3 We present in this chapter different Riemannian surface embedding results: Lagrangian embedding, isometric embedding, and isometric Lagrangian embedding of Riemannian surfaces. The goal is to present important geometric applications to the two previous chapters (in particular the Cartan–Kähler theory), and also to familiarize the reader with techniques used in the next chapters for understanding and solving the generalized isometric embedding problem of vector bundles. This chapter has a sub-appendix where we present and prove the embedding result that can be generalized in higher dimensions, i.e., the existence of Lagrangian manifolds and the Cartan–Janet theorem.

Chapter4 We state the generalized isometric embedding problem of vector bundles, and show its links, on one hand, with the (classic) isometric embedding problem of Riemannian manifolds, and, on the other hand, with conservation laws and harmonic maps between Riemannian manifolds. Section 1 is then dedicated to defining conservation laws in terms of tangent vector fields and in terms of differential forms. The generalized isometric embedding problem is stated in section 2. The principal motivations are presented in section 3. First, we present the relation between the classic isometric embedding problem and the generalized one. Then, we define harmonic maps between two Riemannian manifolds. Several examples of harmonic maps are expounded, and we give their relations to the conservation laws and to the generalized isometric embedding problem. In section 4, we gather all of the generalized isometric embedding results obtained in this thesis. Finally, section 5 is dedicated to presenting an application to energy-momentum tensors. A sub-appendix is dedicated to presenting supplemental details for the lemma's proof used in the corollary's proof pertaining to the conservation laws of energy-momentum tensors.

Chapter 5 In this chapter, we explain and establish a strategy to solve the generalized isometric embedding problem in the general case, and treat the conservation laws case, i.e., when $p = m - 1$. This chapter is based upon [Kah08b, Kah09]. In section 1, we translate the generalized isometric embedding problem in terms of an exterior differential system. We define the notion of a generalized torsion, generalized Bianchi identities, generalized curvature tensors space, generalized Cartan identities, and the generalized Gauss map. Section 2 is dedicated to proving Theorem 0.2 whose key point is Lemma 5.14. An additional proof is also presented by explicitly constructing an ordinary integral flag. The (very technical) proof of Lemma 5.14 is presented for several cases in order to ease the comprehension of the technical details.

Chapter 6 In this last chapter, we present other results of the generalized isometric embedding problem: the case of covariantly closed vector bundle valued differential 1-forms, and the case of a 2-form over a differential manifolds of dimension 4 with values in a vector bundle of rank 3 equipped with an anti-self-dual connection. The sub-appendix of this chapter is dedicated to presenting another way of expressing the generalized Bianchi identities for the case $n = 2, m = 3$ and $p = 1$, and also for the case $n = m = 3$ and $p = 1$, by defining a vector-valued cross product.

Appendix 1 Here we gather the proofs and the technical computations for the results stated in the first and second sections of chapter 1. Despite the fact that these results are classic, we included them in this appendix to lighten chapter 1 and also to be complete.

APPENDIX 2 We briefly introduce the linear Pfaffian system theory and the Cartan–Kähler theory via tableaux. We present some applications: the heat equation on $\mathbb{R}^2$, the conformal embedding, and the existence of Lagrangian manifolds in the complex space $\mathbb{C}^m$.

PERSPECTIVES Concerning Theorem 0.2, it is natural to wonder, as in the classical isometric embedding problem of Riemannian manifolds, whether or not the condition of the data's regularity may be weakened (i.e., having the same result with $\mathcal{C}^\infty$ or $\mathcal{C}^k$ data). Moreover, we expect that it is possible to have a global generalized isometric embedding using Gromov's homotopy principal. Evidently, we expect that, in either case, the dimension of the target space would be greater.

The generalized isometric embedding problem remains open for $p = 2, \ldots, m-2$. Moreover, the question arises of the existence of possible obstructions, and in the affirmative case, whether or not an explicit counter-example may be shown. Finally, the case of the covariantly closed vector bundle valued form of non-constant rank remains an open question.

Chapter 1

Cartan's structure equations

The goal of this chapter is to introduce the fundamental notions of differential geometry expressed in moving frames, and to establish Cartan's structure equations. Almost everything revolves around differential forms, and thus is expressed in the Cartan formalism. The first section is dedicated to introducing and defining the notion of a connection and its curvature on an arbitrary vector bundle, where the dimension of the fiber is not necessarily equal to the dimension of the base manifold, and the relationship between the connection and its curvature is given by Cartan's second-structure equation. Also shown is an interesting property of the connection when the vector bundle is endowed with a metric, and, as always in differential geometry, all of the transformation rules for these objects will be demonstrated. In the second section, a special and fundamental class of vector bundles is investigated: the tangent bundle of a differentiable manifold. The notion of torsion of a connection appears and leads to Cartan's first-structure equation. An important technical result, the Cartan lemma, is stated because it is useful in several applications. The proofs from the first and second sections are given in appendix 1 not only because the calculations are not that difficult and almost all of the results derive from the definitions, but also to lighten the reading. Finally, since Cartan's structure equations are "intensely" used from chapters 3 to 6, the modest purpose of section 3 is to provide some useful applications to Cartan's structure equations in the study of surfaces, such as computing Christoffel symbols, computing the Gauss curvature and presenting a problem studied by Poincaré pertaining to the conformal metrics of constant curvature.

1.1 Connection on a vector bundle

Let $\xi = (\mathbb{V}, \pi, \mathcal{M})$ be a vector bundle over a smooth m-dimensional manifold $\mathcal{M}$ with an r-dimensional vector space $\mathbb{V}$ as a standard fiber. Denote by $(\Gamma(\mathrm{T}\mathcal{M}), [,])$ the Lie algebra of vector fields on $\mathcal{M}$ and by $\Gamma(\mathbb{V})$ the moduli space of cross-sections of the vector bundle $\mathbb{V}$.

Definition 1.1 – Connection on a vector bundle A connection on a vector bundle is a $\Gamma(\mathbb{V})$-valued bilinear operator ∇ on $\Gamma(\mathrm{T}\mathcal{M}) \times \Gamma(\mathbb{V})$ which satisfies $\nabla_{(f\mathrm{X})}\mathrm{S} = f\nabla_{\mathrm{X}}\mathrm{S}$ and a Leibniz identity type $\nabla_{\mathrm{X}}(f\mathrm{S}) = \mathrm{X}(f)\mathrm{S} + f\nabla_{\mathrm{X}}\mathrm{S}$ for all $\mathrm{X} \in \Gamma(\mathrm{T}\mathcal{M})$ and for all $\mathrm{S} \in \Gamma(\mathbb{V})$.

A connection on a vector bundle appears to be a way of "differentiating" cross-sections along vector fields in a way that is analogous to the exterior differential of functions.

Definition 1.2 – Curvature of a connection Let ∇ be a connection on ξ. The curvature of ∇ is a $\Gamma(\mathbb{V})$-valued trilinear operator $\mathcal{R}^{\nabla}$ on $\Gamma(\mathrm{T}\mathcal{M}) \times \Gamma(\mathrm{T}\mathcal{M}) \times \Gamma(\mathbb{V})$ which associates any cross-section S and any two vector fields X and Y with the cross section

$$\mathcal{R}^{\nabla}(\mathrm{X},\mathrm{Y})\mathrm{S} = \Big(\nabla_{\mathrm{X}}\nabla_{\mathrm{Y}} - \nabla_{\mathrm{Y}}\nabla_{\mathrm{X}} - \nabla_{[\mathrm{X},\mathrm{Y}]}\Big)\mathrm{S}. \tag{1.1}$$

It follows immediately that $\mathcal{R}^{\nabla}(\mathrm{X},\mathrm{Y})\mathrm{S} = -\mathcal{R}^{\nabla}(\mathrm{Y},\mathrm{X})\mathrm{S}$. From the definition, one can easily check the following property of the curvature of a connection on an arbitrary vector bundle.

Theorem 1.3 – The tensorial nature of the curvature Let ∇ be a connection on a vector bundle $\mathbb{V}$ of rank r over an m-dimensional manifold $\mathcal{M}$. Then, for any f, g and h smooth functions on $\mathcal{M}$, $\mathrm{S} \in \Gamma(\mathrm{E})$ a section of ξ and $\mathrm{X}, \mathrm{Y} \in \Gamma(\mathrm{T}\mathcal{M})$ two tangent vector fields of $\mathcal{M}$, we have:

$$\mathcal{R}^{\nabla}(f\mathrm{X}, g\mathrm{Y})(h\mathrm{S}) = f.g.h.\mathcal{R}^{\nabla}(\mathrm{X},\mathrm{Y})\mathrm{S}. \tag{1.2}$$

Depending on the situation, expressing the connection and its curvature following Cartan's formalism seems to be more convenient. For that purpose, let us define a flexible generalization of the notion of a frame which seems to be very useful and more adequate in the study of extrinsic geometry of embedded submanifolds.

Definition 1.4 – Moving frame Denote by $\mathcal{O}$ an open subset of $\mathcal{M}$. A set of r local sections $\mathrm{S} = (\mathrm{S}_1, \mathrm{S}_2, \dots, \mathrm{S}_r)$ of ξ is called a moving frame (or a frame field) if for all p in $\mathcal{O}$, $\mathrm{S}(p) = \Big(\mathrm{S}_1(p), \mathrm{S}_2(p), \dots, \mathrm{S}_r(p)\Big)$ forms a basis of the fiber $\mathbb{V}_p$ over the point p.

In the mid 19$^{\text{th}}$ century, Frenet and Serret were pioneer in using moving frames in the study of curves in a 3-dimensional Euclidean space. Later, Darboux studied the problem of constructing a preferred moving frame on a surface in a Euclidean space, and it turned out to be impossible in general to construct such a frame because there were integrability conditions which needed to be satisfied. The definition of a moving frame was developed in the beginning of the 20$^{\text{th}}$ century by Élie Cartan in the study of submanifolds in more general homogeneous spaces, and he formulated and applied "*la méthode du repère mobile*".

If we consider $\mathrm{X} \in \Gamma(\mathrm{T}\mathcal{M})$ to be a tangent vector field on $\mathcal{M}$, then since $\nabla_{\mathrm{X}}\mathrm{S}_j$ is another section of ξ, it can be expressed in the moving frame S as follows:

$$\nabla_{\mathrm{X}}\mathrm{S}_j = \sum_{i=1}^{r} \omega_j^i(\mathrm{X})\mathrm{S}_i \tag{1.3}$$

where $\omega_j^i \in \Gamma(\mathrm{T}^*\mathcal{M})$ are differential 1-forms on $\mathcal{M}$.

Definition 1.5 – Connection 1-form The $r \times r$ matrix $\omega = (\omega_j^i)$ whose entries are differential 1-forms is called the connection 1-form of ∇.

The connection ∇ is completely determined by the matrix $\omega = (\omega_j^i)$. Conversely, a matrix of differential 1-forms on $\mathcal{M}$ determines a connection (in a non-invariant way depending on the choice of the moving frame).

Let $\mathrm{X}, \mathrm{Y} \in \Gamma(\mathrm{T}\mathcal{M})$ be two tangent vector fields. As previously, since $\mathcal{R}^{\nabla}(\mathrm{X},\mathrm{Y})\mathrm{S}_j$ is a section of ξ, it can be expressed on the moving frame S as follows:

$$\mathcal{R}^{\nabla}(\mathrm{X},\mathrm{Y})\mathrm{S}_j = \sum_{i=1}^{r} \Omega_j^i(\mathrm{X},\mathrm{Y})\mathrm{S}_i \tag{1.4}$$

where $\Omega_j^i \in \Gamma(\wedge^2\mathrm{T}^*\mathcal{M})$ are differential 2-forms on $\mathcal{M}$.

Definition 1.6 – Curvature 2-form of a connection The $r \times r$ matrix $\Omega = (\Omega_j^i)$, whose entries are differential 2-forms, is called the curvature 2-form of the connection ∇.

With this viewpoint, we can state the following theorem that gives the relation between the connection 1-form ω and the curvature 2-form Ω.

Theorem 1.7 – Cartan's second-structure equation Let ∇ be a connection on a vector bundle $(\mathbb{V}, \pi, \mathcal{M})$ of rank r over an m-dimensional manifold. Denote by $\omega = (\omega^i_j)$ the $\mathfrak{gl}(r;\mathbb{R})$ valued differential 1-form of the connection ∇. Then

$$\mathrm{d}\omega^i_j + \omega^i_k \wedge \omega^k_j = \Omega^i_j \quad \text{for all } i, j. \tag{1.5}$$

A condensed way to write the Cartan's second-structure equation is

$$\mathrm{d}\omega + \omega \wedge \omega = \Omega \tag{1.6}$$

and by exterior differentiation, we establish the Bianchi identities as follows

Proposition 1.8 – Bianchi identities via differential forms Let ∇ be a connection on ξ. Denote by ω and Ω the connection 1-form and the curvature 2-form of the connection ∇ respectively. Then the expression of the Bianchi identities via differential forms is

$$\mathrm{d}\Omega = \Omega \wedge \omega - \omega \wedge \Omega. \tag{1.7}$$

The expressions of the differential 1-form of the connection and the curvature 2-form are both local. As always in differential geometry, one should know how these expressions are changed in another coordinate system.

Proposition 1.9 – Connection and curvature transformation rules Let ∇ be a connection on a vector bundle $(\mathbb{V}, \pi, \mathcal{M})$ of rank r over an m-dimensional manifold. Let $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$ be two neighborhoods of a point $\mathrm{M} \in \mathcal{M}$. Consider $\varphi_\alpha : \pi^{-1}(\mathcal{O}_\alpha) \longrightarrow \mathcal{O}_\alpha \times \mathbb{R}^r$ and $\varphi_\beta : \pi^{-1}(\mathcal{O}_\beta) \longrightarrow \mathcal{O}_\beta \times \mathcal{R}^r$. The transition map is $g_{\alpha\beta} : \mathcal{O}_\alpha \cap \mathcal{O}_\beta \longrightarrow \mathrm{GL}(n;\mathbb{R}^r)$. Denote by $\omega(\alpha)$ and $\omega(\beta)$ the expressions of the connection 1-form of ∇ on $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$ respectively. Denote by $\Omega(\alpha)$ and $\Omega(\beta)$ the expressions of the curvature 2-form of ∇ on $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$ respectively. Then

$$\omega(\beta) = g^{-1}_{\alpha\beta}\mathrm{d}g_{\alpha\beta} + g^{-1}_{\alpha\beta}\omega(\alpha)g_{\alpha\beta} \tag{1.8}$$

$$\Omega(\beta) = g^{-1}_{\alpha\beta}\Omega(\alpha)g_{\alpha\beta} \tag{1.9}$$

An interesting property for a connection, when the vector bundle is endowed with a Riemannian metric, is to be "compatible" with that metric. Let us recall that a Riemannian metric g on ξ is a positively-defined scalar product on each fiber.

Definition 1.10 – Metric connection A connection ∇ on a vector bundle endowed with a Riemannian metric is said to be compatible with the metric g, or simply, to be a metric connection, if ∇ satisfies the Leibniz identity

$$\nabla_{\mathrm{X}}\Big(g(\mathrm{S}_1, \mathrm{S}_2)\Big) = g(\nabla_{\mathrm{X}}\mathrm{S}_1, \mathrm{S}_2) + g(\mathrm{S}_1, \nabla_{\mathrm{X}}\mathrm{S}_2), \quad \forall \mathrm{S}_1, \mathrm{S}_2 \in \Gamma(\mathbb{V}) \text{ and } \forall \mathrm{X} \in \Gamma(\mathrm{T}\mathcal{M}). \tag{1.10}$$

The following result shows an interesting property of the metric connections and will be useful for many applications in chapter 3 and for the generalized isometric embedding problem.

Proposition 1.11 — Connection and curvature forms of a metric connection Let $\mathrm{S} = (\mathrm{S}_1, \mathrm{S}_2, \ldots, \mathrm{S}_n)$ be an orthonormal moving frame with respect to g, i.e. $g_p(\mathrm{S}_i, \mathrm{S}_j) = \delta_{ij}$ for all $p \in \mathcal{O}$, $i, j = 1, \ldots, r$. The matrix of 1-forms ω associated with S and the curvature matrix of 2-forms Ω are then both skew-symmetric, i.e., $\omega^i_j + \omega^j_i = 0$ and $\Omega^i_j + \Omega^j_i = 0$.

This means that metric connections and their curvatures are $\mathfrak{o}(n)$-valued differential forms rather than $\mathfrak{gl}(n)$-valued differential forms.

1.2 The tangent bundle case

The first vector bundle that probably every student discovers in a differential geometry class is the tangent bundle of a differentiable manifold. We consider in this subsection the tangent bundle of a manifold, i.e., $\mathbb{V} = \mathrm{T}\mathcal{M}$. This class of vector bundles provides more notions. As a special type of vector bundles, all the results stated above obviously remain true. For instance, let us consider, as previous, a local moving frame $\mathrm{S} = (\mathrm{S}_1, \ldots, \mathrm{S}_m)$ over $\mathcal{O} \subset \mathcal{M}$. One can naturally associate S with a moving coframe $\eta = (\eta^1, \ldots, \eta^m)$ defined as a local frame field of 1-forms, such that for all $p \in \mathcal{U}$, $\eta^i(p)(\mathrm{S}_j) = \delta^i_j$.

Definition 1.12 — Torsion of a connection Let $\mathcal{M}$ be an m-dimensional manifold and ∇ be a connection on the tangent bundle $\mathrm{T}\mathcal{M}$. The torsion of ∇ is a $\Gamma(\mathrm{T}\mathcal{M})$-valued bilinear operator T^∇ on $\Gamma(\mathrm{T}\mathcal{M}) \times \Gamma(\mathrm{T}\mathcal{M})$ which associates any two vector fields X and Y with the vector field:

$$\mathrm{T}^\nabla(\mathrm{X}, \mathrm{Y}) := \nabla_\mathrm{X}\mathrm{Y} - \nabla_\mathrm{Y}\mathrm{X} - [\mathrm{X}, \mathrm{Y}] \tag{1.11}$$

As for the connection and the curvature, we express the notion of torsion in moving frames as follows:

Definition 1.13 — Torsion 2-form of a connection The torsion 2-form of a connection ∇ is a $\mathrm{T}\mathcal{M}$-valued differential 2-form $\Theta = (\Theta^i)$ such that $\Theta^i := \mathrm{d}\eta^i + \omega^i_j \wedge \eta^j$. Moreover, if $\Theta = 0$, then the connection is said to be torsion-free.

The torsion 2-form can be written in a more condensed way and is sometimes called Cartan's first-structure equation in mathematical literature :

$$\mathrm{d}\eta + \omega \wedge \eta = \Theta. \tag{1.12}$$

On a differentiable manifold equipped with a connection, the torsion of a connection measures the default for a connection to have a parallelogram property. A torsion-free connection which is also compatible with a Riemannian metric is said to be a Levi-Civita connection. Note that a Levi-Civita connection on a Riemannian manifold exists and is unique.

As for the connection 1-form and the curvature 2-form, the following proposition shows how the local expression of the torsion changes in a different local coordinates.

Proposition 1.14 — Torsion transformation rule Let ∇ be a connection on an m-dimensional Riemannian manifold $(\mathcal{M}, g)$. Let $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$ be two neighborhoods of a point $\mathrm{M} \in \mathcal{M}$. Let us consider $\varphi_\alpha : \pi^{-1}(\mathcal{O}_\alpha) \longrightarrow \mathcal{O}_\alpha \times \mathbb{R}^m$ and $\varphi_\beta : \pi^{-1}(\mathcal{O}_\beta) \longrightarrow \mathcal{O}_\beta \times \mathbb{R}^m$. The transition map is then $g_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \longrightarrow \mathrm{GL}(n; \mathbb{R}^m)$. Denote by $\Theta(\alpha)$ and $\Theta(\beta)$ the expressions of the torsion 2-form on $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$ respectively. Then

$$\Theta(\beta) = g^{-1}_{\alpha\beta}\Theta(\alpha). \tag{1.13}$$

The following proposition shows the relationship between the connection 1-form, the curvature 2-form and the torsion 2-form on a tangent bundle of a differentiable manifold.

Proposition 1.15 – Relationship between the connection, curvature and torsion Let ∇ be a connection on an m-dimensional Riemannian manifold $(\mathcal{M}, g)$. Denote by ω the connection 1-form of ∇, Ω its curvature, Θ its torsion and $\eta = (\eta^1, \ldots, \eta^m)$ a moving coframe. Then the connection, the curvature, torsion and the coframe are related by

$$\mathrm{d}\Theta + \omega \wedge \Theta = \Omega \wedge \eta. \tag{1.14}$$

Finally, the following proposition, which is the purpose of this section, summarizes Cartan's structure equations and the different results obtained above in the case of a Riemannian manifold.

Proposition 1.16 – Cartan's structure-equations for a Riemannian manifold Let $(\mathcal{M}, g)$ be an m-dimensional Riemannian manifold. Let $\eta = (\eta^1, \eta^2, \ldots, \eta^m)$ be an orthonormal moving coframe on $\mathcal{M}$. By equations (1.12), (A.2) and proposition 1.11, we establish the Cartan's structure equations:

$$\begin{cases} \mathrm{d}\eta^i + \omega^i_j \wedge \eta^j = 0 \quad \text{(torsion-free)} \\ \mathrm{d}\omega^i_j + \omega^i_k \wedge \omega^k_j = \Omega^i_j \end{cases} \tag{1.15}$$

where the matrix (ω^i_j) is the Levi-Civita connection 1-form (torsion-free connection which is compatible with the Riemannian metric g). Moreover, since η is an orthonormal coframe field, (ω^i_j) and (Ω^i_j) are skew-symmetric.

We conclude this section with the statement of a technical result which is easy to prove and very useful for the applications.

Lemma 1.17 – Cartan lemma Let $\mathcal{M}$ be an m-dimensional manifold. Let us consider $\omega^1, \omega^2, \ldots, \omega^r$ to be linearly independent differential 1-forms on $\mathcal{M}$, where $r \leqslant n$, and let $\theta^1, \theta^2, \ldots, \theta^r$ be r differential form on $\mathcal{M}$ such that $\sum_{i=1}^{r} \theta^i \wedge \omega^i = 0$. There then exist r^2 functions $\mathcal{C}^\infty$ on $\mathcal{M}$ h^i_j such that $\theta^i = \sum_{j=1}^{r} h^i_j \omega^j$ where $h^i_j = h^j_i$.

1.3 Applications to surfaces

As mentioned previously, beginning in chapter 3, there will be many possibilities of applications of Cartan's structure equations in the proofs and explanations of geometric problems. The modest goal of this section is to present some useful and interesting applications of Cartan's structure equations for Riemannian surfaces. Moreover, we expound the problem of finding a conformal metric for a given Riemannian surface of constant Gauss curvature, a problem completely studied by Poincaré and which has a generalization in higher dimensions and is known as the Yamabe problem.

1.3.1 Christoffel symbols and Gauss curvature

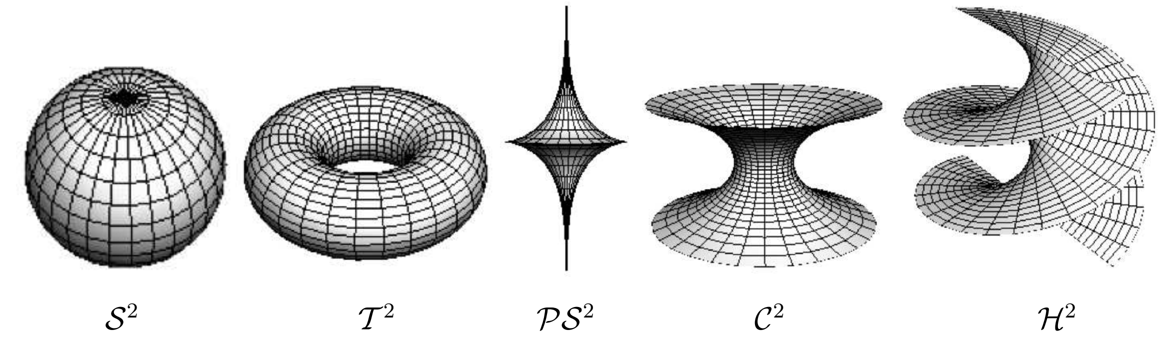

Figure 1.1: Sphere, torus, pseudo-sphere, catenoid and helicoid

Let $u : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be a parameterization of a surface in the 3-dimensional space $\mathbb{R}^3$ endowed with the standard metric. Locally, $u(\theta, \varphi) = \Big(\mathrm{X}(\theta, \varphi), \mathrm{Y}(\theta, \varphi), \mathrm{Z}(\theta, \varphi)\Big)$.

Examples 1.18 – Parameterization of surfaces.

1. **The sphere $\mathcal{S}^2$:** $(\theta, \varphi) \longrightarrow (\cos\theta\cos\varphi, \sin\theta\cos\varphi, \sin\varphi)$.
2. **The torus of revolution $\mathcal{T}^2$:** $(\theta, \varphi) \longrightarrow ((\mathrm{R} + r\cos\varphi)\cos\theta, (\mathrm{R} + r\cos\varphi)\sin\theta, r\sin\varphi)$, where $\mathrm{R} > r > 0$.
3. **The pseudo-sphere $\mathcal{PS}^2$:** $(\theta, \varphi) \longrightarrow (\cos\varphi/\cosh\theta, \sin\varphi/\cosh\theta, \theta - \tanh\theta)$.
4. **The catenoid $\mathcal{C}^2$:** $(\theta, \varphi) \longrightarrow (\cosh\theta\cos\varphi, \cosh\theta\sin\varphi, \theta)$.
5. **The helicoid $\mathcal{H}^2$:** $(\theta, \varphi) \longrightarrow (\theta\cos\varphi, \theta\sin\varphi, \varphi)$.

Thus the pull-back of the Euclidean metric $\langle, \rangle_{\mathbb{R}^3} = \mathrm{dX} \otimes \mathrm{dX} + \mathrm{dY} \otimes \mathrm{dY} + \mathrm{dZ} \otimes \mathrm{dZ}$ by u is

$$u^*(\langle, \rangle_{\mathbb{R}^3}) = \Big(\mathrm{X}_\theta^2 + \mathrm{Y}_\theta^2 + \mathrm{Z}_\theta^2\Big)\mathrm{d}\theta\otimes\mathrm{d}\theta + \Big(\mathrm{X}_\theta\mathrm{X}_\varphi + \mathrm{Y}_\theta\mathrm{Y}_\varphi + \mathrm{Z}_\theta\mathrm{Z}_\varphi\Big)(\mathrm{d}\theta\otimes\mathrm{d}\varphi + \mathrm{d}\varphi\otimes\mathrm{d}\theta) + \Big(\mathrm{X}_\varphi^2 + \mathrm{Y}_\varphi^2 + \mathrm{Z}_\varphi^2\Big)\mathrm{d}\theta\otimes\mathrm{d}\theta$$

where X_θ and X_φ are the partial derivatives of the function X with respect to θ and φ respectively.

Examples 1.19 – Metrics on surfaces. The metric expressions for the above surfaces are:

1. **The Sphere:** $g_{\mathcal{S}^2} = \cos^2\varphi \mathrm{d}\theta \otimes \mathrm{d}\theta + \mathrm{d}\varphi \otimes \mathrm{d}\varphi$
2. **The torus of revolution:** $g_{\mathcal{T}^2} = (\mathrm{R} + r\cos\varphi)^2 \mathrm{d}\theta \otimes \mathrm{d}\theta + r^2 \mathrm{d}\varphi \otimes \mathrm{d}\varphi$.
3. **The pseudo-sphere:** $g_{\mathcal{PS}^2_{\mathrm{R}}} = \mathrm{R}^2\varphi(\mathrm{d}\theta \otimes \mathrm{d}\theta + \sinh^2\varphi \mathrm{d}\varphi \otimes \mathrm{d}\varphi)/\cosh\varphi^2$.
4. **The catenoid:** $g_{\mathcal{C}^2} = \cosh^2\theta(\mathrm{d}\theta \otimes \mathrm{d}\theta + \mathrm{d}\varphi \otimes \mathrm{d}\varphi)$.
5. **The helicoid:** $g_{\mathcal{H}^2} = \mathrm{d}\theta \otimes \mathrm{d}\theta + (1 + \theta^2)\mathrm{d}\varphi \otimes \mathrm{d}\varphi$.

The following proposition[1] gives three different models of an important manifold: the hyperbolic space.

Proposition 1.20 – Three isometric models for the 3-hyperbolic space The following three Riemannian manifolds are isometric:

1. **The hyperboloid upper-sheet:** $(\mathcal{HU}^2, g_{\mathcal{HU}^2})$ where $\mathcal{HU}^2 = \{(x,y,z) \in \mathbb{R}^3 | z^2 - x^2 - y^2 = 1 \text{ and } z > 0\}$ and $g_{\mathcal{HU}^2} = \iota^*(\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y - \mathrm{d}z \otimes \mathrm{d}z)$, ι is the canonical injection of $\mathcal{HU}^2$ in $\mathbb{R}^3$.
2. **The Poincaré ball:** $(\mathcal{PB}^2, g_{\mathcal{PB}^2})$ where $\mathcal{PB}^2 = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 < 1\}$, which is the usual ball of radius 1 in $\mathbb{R}^2$ and $g_{\mathcal{PB}^2} = 4(\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y)/(1 - x^2 - y^2)^2$.
3. **The Poincaré half-space:**$(\mathcal{PH}^2, g_{\mathcal{PH}^2})$ where $\mathcal{PH}^2 = \{(x,y) \in \mathbb{R}^2 | y > 0\}$ and the metric $g_{\mathcal{PH}^2} = (\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y)/y^2$.

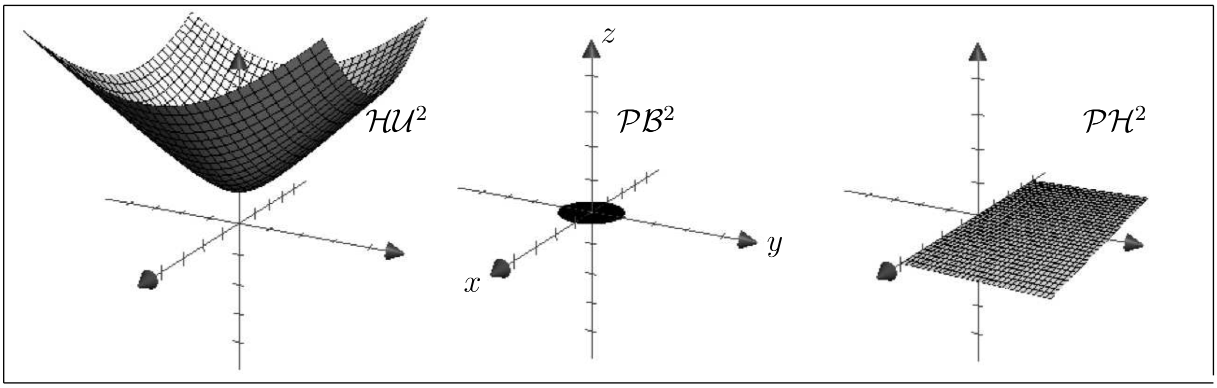

Figure 1.2: Hyperbolic space models.

In an integrable moving frame, i.e. such that $[\mathrm{S}_i, \mathrm{S}_j] = 0$ for all i, j, or equivalently, on a coordinate system, the Christoffel symbols can be computed as follows:

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \Big(\partial_i g_{lj} + \partial_j g_{il} - \partial_l g_{ij} \Big). \tag{1.16}$$

To compute the Christoffel symbols for a surface, we need to invert the metric and compute the derivative in both directions of the coordinate system, then we apply the formula. It seems to be much easier to compute the 2^3 Christoffel symbols by taking advantage of Cartan's structure equations and the skew-symmetry of the Levi-Civita connection. Denote by (η^1, η^2) an orthonormal moving coframe and by $(\mathrm{S}_1, \mathrm{S}_2)$ its dual moving frame. The Christoffel symbols can be expressed as follows

$$\Gamma_{ij}^k = \omega_j^k(\mathrm{S}_i). \tag{1.17}$$

and hence

$$\omega_j^k = \Gamma_{ij}^k \eta^i \tag{1.18}$$

Since the connection is a $\mathfrak{o}(2)$-valued differential 1-form,

$$\Gamma_{i1}^1 = \Gamma_{i2}^2 = 0 \quad \text{ and } \Gamma_{i2}^1 + \Gamma_{i1}^2 = 0. \tag{1.19}$$

[1]See [Kah05] for the proof.

Proposition 1.21 – Connection 1-form and Gauss curvature of a surface Let $(\mathcal{M}^2, g)$ be a Riemannian surface such that the metric $g = a^2 \mathrm{d}\theta \otimes \theta + b^2 \mathrm{d}\varphi \otimes \mathrm{d}\varphi$ in an orthonormal moving coframe, where a and b are functions. Then the connection 1-form (ω^i_j) of the Levi-Civita connection and the Gauss curvature κ of $(\mathcal{M}^2, g)$ are:

$$\omega^1_2 = \frac{a_\varphi}{b}\mathrm{d}\theta - \frac{b_\theta}{a}\mathrm{d}\varphi \qquad \text{and} \qquad \omega^1_2 + \omega^2_1 = \omega^1_1 = \omega^2_2 = 0 \tag{1.20}$$

$$\mathcal{K}_g = -\frac{1}{ab}\left(\frac{b_{\theta\theta}}{a} + \frac{a_{\varphi\varphi}}{b} - \frac{a_\theta b_\theta}{a^2} - \frac{a_\varphi b_\varphi}{b^2}\right). \tag{1.21}$$

Proof. The g-orthonormal coframe of the surface is: $\eta^1 = a\mathrm{d}\theta$ and $\eta^2 = b\mathrm{d}\varphi$, where a and b are non-vanishing functions. On one hand, $\mathrm{d}\eta^1 = a_\varphi \mathrm{d}\varphi \wedge \mathrm{d}\theta = -(a_\varphi/b)\mathrm{d}\theta \wedge \eta^2$ and on the other hand, $\mathrm{d}\eta^2 = b_\theta \mathrm{d}\theta \wedge \mathrm{d}\varphi = -(b_\theta/a)\mathrm{d}\varphi \wedge \eta^1$. Consequently, by Cartan's first-structure equation, which expresses the torison-free of the Levi-Civita connection,

$$\omega^1_2 = \frac{a_\varphi}{b}\mathrm{d}\theta - \frac{b_\theta}{a}\mathrm{d}\varphi \tag{1.22}$$

Recall that the skew-symmetry of the connection 1-form is due to the fact that the Levi-Civita connection is compatible with the metric and hence is an $\mathfrak{o}(2)$-valued differential 1-form. Finally, by computing the exterior derivative $\mathrm{d}\omega^1_2$ and by using Cartan's second-structure equation, we find the expression of the Gauss equation, as in (1.21). □

We summarize in table 1.1 the coframe expression, the non-vanishing term of the connection 1-form, the Christoffel symbols and the Gauss curvature of both the surfaces given in the examples and for two models of the hyperbolic space. The following proposition shows the new expression of the Gauss curvature when the metric is multiplied by a positive real number. For instance it allows us to know the Gauss curvature of a sphere of radius R when we know the Gauss curvature of the unit sphere. The proof is a special case of Proposition 1.23.

Proposition 1.22 – Gauss curvature for dilated metrics Let $(\mathcal{M}^2, g_0)$ be a Riemannian surface and let $\mathcal{K}_{g_0}$ be its Gauss curvature. If $\mathrm{R} \neq 0$ is a real number, then the Gauss curvature of $(\mathcal{M}^2, g)$, where $g = \mathrm{R}^2 g_0$, is $\mathcal{K}_g = \mathcal{K}_{g_0}/\mathrm{R}^2$.

1.3.2 Existence of conformal metrics with constant Gauss curvature

In the well-known problem studied by Poincaré, we are interested in finding out whether or not a Riemannian surface is conformally equivalent to a constant Gauss curvature Riemannian surface. The problem is generalized in higher dimensions by replacing the Gauss curvature with the scalar curvature. This is known as the Yamabe problem. Let $(\mathcal{M}, g_0)$ be a Riemannian surface. As shown previously, one can consider an orthonormal moving coframe on which the metric is expressed locally as: $g_0 = \eta^1 \otimes \eta^1 + \eta^2 \otimes \eta^2$. For an arbitrary metric g_0, the Gauss curvature has a higher chance to be non-constant. We also know that multiplying the metric by a positive number R^2 changes the Gauss curvature form $\mathcal{K}$ to $\mathcal{K}/\mathrm{R}^2$, and hence, a non-constant Gauss curvature remains non-constant by just dilating the metric, however, one can imagine that multiplying the metric by a non-vanishing function can neutralize the variation of the Gauss curvature. So, is it possible to find a metric g, conformally equivalent to g_0, such that

$(\mathcal{M}, g)$ is of constant Gauss curvature? Namely, is it possible to choose a function λ such that the Riemannian surface $(\mathcal{M}, g)$, where $g = e^{2\lambda} g_0$, is of constant Gauss curvature?

Proposition 1.23 – Prescribed Gauss curvature Let $(\mathcal{M}, g_0)$ be a Riemannian surface. Then the Gauss curvature $\mathcal{K}_g$ of $(\mathcal{M}, g)$, where $g = e^{2\lambda} g_0$, satisfies the following equation:

$$\mathcal{K}_g \omega^1 \wedge \omega^2 = \mathcal{K}_{g_0} \eta^1 \wedge \eta^2 + \mathrm{d} * \mathrm{d}\lambda \tag{1.23}$$

where $\omega^1 = e^{\lambda}\eta^1$ and $\omega^2 = e^{\lambda} g$

In particular, to find a conformal metric on $\mathcal{M}$ with a constant Gauss curvature, one needs to find the function λ solution to the equation 1.23, where $\mathcal{K}_g$ is constant.

Proof. Let (η^1, η^2) be an orthonormal coframe such that $g_0 = \eta^1 \otimes \eta^1 + \eta^2 \otimes \eta^2$. Let $g = e^{2\lambda} g_0$, where λ is a function on $\mathcal{M}$. Let us then denote $\omega^1 = e^{\lambda}\eta^1$ and $\omega^2 = e^{\lambda} g$. Since e^{λ} does not vanish, (ω^1, ω^2) is a moving coframe on $\mathcal{M}$. On one hand,

$$\mathrm{d}\omega^1 = \mathrm{d}(e^{\lambda}\eta^1) = \mathrm{d}(e^{\lambda}) \wedge \eta^1 + e^{\lambda}\mathrm{d}\eta^1 = e^{\lambda}\mathrm{d}\lambda \wedge \eta^1 - e^{\lambda}(\eta^1_2 \wedge \eta^2) = \mathrm{d}\lambda \wedge \omega^1 - \eta^1_2 \wedge \omega^2 \tag{1.24}$$
$$\mathrm{d}\omega^2 = \mathrm{d}(e^{\lambda}\eta^2) = \mathrm{d}(e^{\lambda}) \wedge \eta^2 + e^{\lambda}\mathrm{d}\eta^2 = e^{\lambda}\mathrm{d}\lambda \wedge \eta^2 + e^{\lambda}(\eta^1_2 \wedge \eta^1) = \mathrm{d}\lambda \wedge \omega^2 + \eta^1_2 \wedge \omega^1 \tag{1.25}$$

where (η^i_j) is the Levi-Civita connection 1-form on $(\mathcal{M}, g_0)$. On the other hand, $g = \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2$ and hence, the coframe (ω^1, ω^2), which is orthonormal for the metric g, satisfies Cartan's structure equations. Consequently,

$$\mathrm{d}\omega^1 + \omega^1_2 \wedge \omega^2 = 0 \quad \text{and} \quad \mathrm{d}\omega^2 - \omega^1_2 \wedge \omega^1 = 0 \tag{1.26}$$

where (ω^i_j) is the Levi-Civita connection 1-form for $(\mathcal{M}, g)$. Therefore

$$\mathrm{d}\lambda \wedge \omega^1 + (\omega^1_2 - \eta^1_2) \wedge \omega^2 = 0 \tag{1.27}$$
$$(\omega^1_2 - \eta^1_2) \wedge \omega^1 - \mathrm{d}\lambda \wedge \omega^2 = 0. \tag{1.28}$$

Applying the Cartan lemma for (1.27) yields to $\mathrm{d}\lambda = a\omega^1 + b\omega^2$ and $(\omega^1_2 - \eta^1_2) = b\omega^1 + c\omega^2$, where a, b and c are functions on $\mathcal{M}$. However, the equation (1.28) imposed that $a + c = 0$, and hence

$$(\omega^1_2 - \eta^1_2) = *\mathrm{d}\lambda. \tag{1.29}$$

By the exterior differentiation of (1.29) and by using Cartan's second-structure equation, we conclude that $\mathrm{d}\omega^1_2 - \mathrm{d}\eta^1_2 = \kappa_g \omega^1 \wedge \omega^2 - \kappa_{g_0} \eta^1 \wedge \eta^2 = \mathrm{d} * \mathrm{d}\lambda$, where $\mathcal{K}_g$ and $\mathcal{K}_{g_0}$ are the Gauss curvature of the Riemannian surfaces $(\mathcal{M}, g)$ and $(\mathcal{M}, g_0)$ respectively. □

Remarks 1.24

1. If the function λ is constant, then Proposition 1.23 reduces to Proposition 1.22. Indeed, if $\mathrm{R} = e^{\lambda}$, then $\mathrm{d} * \mathrm{d}\lambda$ vanishes and the equation 1.23 becomes $\mathcal{K}_g \omega^1 \wedge \omega^2 - \mathcal{K}_{g_0} \eta^1 \wedge \eta^2 = \mathcal{K}_g \mathrm{R}\eta^1 \wedge \mathrm{R}\eta^2 - \mathcal{K}_{g_0} \eta^1 \wedge \eta^2 = (\mathrm{R}^2 \mathcal{K}_g - \mathcal{K}_{g_0}) \eta^1 \wedge \eta^2 = 0$, and thus $\mathrm{R}^2 \mathcal{K}_g = \mathcal{K}_{g_0}$.

2. If $\mathcal{M}^2 = \mathbb{R}^2$ and g is conformal to the standard Euclidean metric on $\mathbb{R}^2$, i.e., $g = e^{2\lambda} \langle , \rangle_{\mathbb{R}^2}$ then since the Gauss curvature of $(\mathbb{R}^2, \langle , \rangle_{\mathbb{R}^2})$ vanishes, the equation 1.23 reduces to

$$\mathcal{K}_g = e^{-2\lambda} \Delta\lambda. \tag{1.30}$$

3. If $\mathcal{M}$ is a compact surface without boundary, then integrating equation 1.23 leads to

$$\int_{\mathcal{M}} \mathcal{K}_g \omega^1 \wedge \omega^2 = \int_{\mathcal{M}} \mathcal{K}_{g_0} \eta^1 \wedge \eta^2 \tag{1.31}$$

where $\int \mathrm{d} * \mathrm{d}\lambda = 0$ by the Stokes theorem. Moreover, the Gauss-Bonnet formula assures that the integral of the Gauss curvature is equal to $2\pi\chi(\mathcal{M})$, where $\chi(\mathcal{M})$ is the Euler characteristic associated to the topological space $\mathcal{M}$. Therefore, if $\mathcal{K}_g$ is constant, then equation 1.23 has no solution if $\mathcal{K}_g$ and $\chi(\mathcal{M})$ are of opposite signs.

Taking advantage of Proposition 1.23 proof, we present another way to compute the Gauss curvature of a given Riemannian by using Cartan second-structure equations:

Proposition 1.25 – Gauss curvature Let $(\mathcal{M}^2, g)$ be a Riemannian surface. Denote by (η^1, η^2) the moving coframe where the metric g is diagonal, i.e., $g = \eta^1 \otimes \eta^1 + \eta^2 \otimes \eta^2$, and denote by $(\mathrm{S}_1, \mathrm{S}_2)$ its dual moving frame. Then the Gauss curvature $\mathcal{K}_g$ of $(\mathcal{M}^2, g)$ is:

$$\mathcal{K}_g = \mathrm{S}_1(\eta^1_2(\mathrm{S}_2)) - \mathrm{S}_2(\eta^1_2(\mathrm{S}_1)) - \eta^1_2([\mathrm{S}_1, \mathrm{S}_2]) \tag{1.32}$$

where η^1_2 is the non-vanishing term of the connection 1-form of the Levi-Civita connection.

Proof. By Cartan second-structure equation, $\mathrm{d}\eta^1_2 = \mathcal{K}_g \eta^1 \wedge \eta^2$. Thus, $\mathcal{K}_g = \mathrm{d}\eta^1_2(\mathrm{S}_1, \mathrm{S}_2)$, and by the Cartan formula, $\mathcal{K}_g = \mathrm{S}_1(\eta^1_2(\mathrm{S}_2)) - \mathrm{S}_2(\eta^1_2(\mathrm{S}_1)) - \eta^1_2([\mathrm{S}_1, \mathrm{S}_2])$. □

$\mathcal{M}^2$	η^1	η^2	ω^1_2	Γ^1_{12}	Γ^1_{22}	$\mathcal{K}_g$
$\mathbb{R}^2$	$\mathrm{d}\theta$	$\mathrm{d}\varphi$	0	0	0	0
$\mathcal{S}^2$	$\cos\varphi \mathrm{d}\theta$	$\mathrm{d}\theta$	$-\sin\varphi \mathrm{d}\theta$	$-\tan\varphi$	0	1
$\mathcal{T}^2$	$(\mathrm{R} + r\cos\varphi)\mathrm{d}\theta$	$r\mathrm{d}\varphi$	$-\sin\varphi \mathrm{d}\theta$	$\dfrac{\sin\varphi}{(\mathrm{R} + r\cos\varphi)}$	0	$\dfrac{\cos\varphi}{r(\mathrm{R} + r\cos\varphi)}$
$\mathcal{PS}^2$	$\dfrac{\mathrm{d}\theta}{\cosh\varphi}$	$\tanh\varphi \mathrm{d}\varphi$	$-\dfrac{\mathrm{d}\theta}{\cosh\varphi}$	-1	0	-1
$\mathcal{H}^2$	$\mathrm{d}\theta$	$\sqrt{1+\theta^2}\mathrm{d}\varphi$	$-\dfrac{\theta \mathrm{d}\varphi}{\sqrt{(1+\theta)}}$	0	$\dfrac{-\theta}{(1+\theta^2)}$	$-\dfrac{1}{(1+\theta^2)^2}$
$\mathcal{C}^2$	$\cosh\theta \mathrm{d}\theta$	$\cosh\theta \mathrm{d}\varphi$	$-\tanh\theta \mathrm{d}\varphi$	0	$\dfrac{-\sinh\theta}{\cosh^4\theta}$	$-\dfrac{1}{\cosh^4\theta}$
$\mathcal{PB}^2$	$\dfrac{2\mathrm{d}x}{(1-x^2-y^2)}$	$\dfrac{2\mathrm{d}y}{(1-x^2-y^2)}$	$\dfrac{2(y\mathrm{d}x - x\mathrm{d}y)}{(1-x^2-y^2)}$	y	$-x$	-1
$\mathcal{PH}^2$	$\dfrac{\mathrm{d}x}{y}$	$\dfrac{\mathrm{d}y}{y}$	$-\dfrac{\mathrm{d}x}{y}$	-1	0	-1

Table 1.1: Coframe, connection 1-form, Christoffel symbols, and Gauss curvature of some surfaces.

Chapter 2

EDS and Cartan–Kähler theory

The reader will notice that exterior differential systems play a central role in this chapter as well as those that follow. Why? What make EDSs so special? Actually, an exterior differential system is nothing but a geometric way of studying a PDE. Indeed, any PDE or system of PDEs represents an exterior differential system on a certain space, and conversely. Defined in section 1 are exterior differential systems on a manifold, exterior ideals and exterior differential ideals. Solving an exterior differential system means finding integral manifolds for the exterior differential ideal generated by it, which represents the equivalent of finding solutions to a PDE. For the case of Pfaffian systems, the necessary and sufficient condition is provided by the Frobenius theorem. However, in many geometric and analytic problems, the exterior differential systems that arise are not always Pfaffian systems, but rather for instance, Lagrangian manifolds for a symplectic manifold. Therefore, section 2 is a brief introduction to the Cartan–Kähler theory [Car71, Käh34], which is a general method for finding and constructing integral manifolds. We start by defining an integral element of an exterior differential system of a given dimension, then we define its polar space and the extension rank, which are of great importance. The successive extensions give an integral flag and all of the technical results that follow are dedicated to both checking the involution and to showing the existence of integral manifolds. The Cartan test is stated to check the involution of an EDS, and to do that, a technical proposition is stated to provide a way of computing the Cartan characters. Then, for the existence of integral manifolds, the Cauchy–Kowaleskaya theorem and the Cartan–Kähler theorem are given.

2.1 Exterior differential systems

Denote by $\Gamma(\wedge \mathrm{T}^*\mathcal{M})$ the space of smooth differential forms on $\mathcal{M}$. This is a graded algebra under the wedge product. We do not use the standard notation $\Omega(\mathcal{M})$ so as to not confuse it with the curvature 2-form of the connection.

Definition 2.1 – EDS Let $\mathcal{M}^m$ be an m-dimensional manifold. An exterior differential system on $\mathcal{M}$ is a finite set of differential forms $\mathrm{I} = \{\omega^1, \omega^2, \dots, \omega^k\} \subset \Gamma(\wedge \mathrm{T}^*\mathcal{M})$ with which the set of equations $\{\omega^i = 0 \,|\, \omega^i \in \mathrm{I}\}$ is associated.

Definition 2.2 – Pfaffian system Let $\mathcal{M}$ be an m-dimensional manifold. A Pfaffian system on $\mathcal{M}$ is an exterior differential system I on $\mathcal{M}$ which contains only linearly independent differential 1-forms.

Examples 2.3 – EDS on $\mathbb{R}^3$.

1. **Pfaffian form:** Let $\mathrm{I}_1 = \{a\mathrm{d}x + b\mathrm{d}y + c\mathrm{d}z\}$ be an EDS on $\mathbb{R}^3$, where a, b and c are functions on $\mathbb{R}^3$. The EDS I is said to be a Pfaffian system because it contains only one differential 1-form.

2. **"Non-Pfaffian" system:** Let $\mathrm{I}_2 = \{\mathrm{d}x - \mathrm{d}y, \mathrm{d}z \wedge \mathrm{d}y\}$ be an EDS on $\mathbb{R}^3$. I_2 is not a Pfaffian system because it contains a differential 2-form.

Definition 2.4 — Exterior Ideal Let $\mathcal{M}$ be an m-dimensional manifold. An exterior ideal is a subset of differential forms $\mathcal{I} \subset \Gamma(\wedge \mathrm{T}^*\mathcal{M})$ such that the exterior product of any differential form of $\mathcal{I}$ by a differential form on $\mathcal{M}$ belongs to $\mathcal{I}$, and if the sum of any two differential forms of the same degree belonging to $\mathcal{I}$, belongs also to $\mathcal{I}$.

Definition 2.5 — Exterior ideal generated by an EDS Let $\mathcal{M}$ be a m-dimensional manifold and I an exterior differential system on $\mathcal{M}$. The exterior ideal generated by I is the smallest exterior ideal containing I.

Examples 2.6 — Exterior ideal generated by an EDS in $\mathbb{R}^3$-Continued.

1. **Pfaffian form:** Denote by ω the differential form of I_1 defined in examples 2.3. Then the exterior ideal generated by I_1 is:

$$\mathcal{I}_1 = \{\omega\}_{\text{alg}} = \{\alpha \wedge \omega | \alpha \in \Gamma(\wedge \mathrm{T}^*\mathbb{R}^3)\} \tag{2.1}$$

2. **"Non-Pfaffian" system:** Denote by ω^1 and ω^2 the differential forms of I_2 defined in examples 2.3. The exterior ideal generated by I_2 is:

$$\mathcal{I}_2 = \{\omega^1, \omega^2\}_{\text{alg}} = \{\alpha \wedge \omega^1 + \beta \wedge \omega^2 | \alpha, \beta \in \Gamma(\wedge \mathrm{T}^*\mathbb{R}^3)\} \tag{2.2}$$

Definition 2.7 — Exterior differential ideal Let $\mathcal{M}$ be an m-dimensional manifold. An exterior differential ideal $\mathcal{I}$ on $\mathcal{M}$ is an exterior ideal which is closed under the exterior differentiation, i.e., $\mathrm{d}\mathcal{I} \subset \mathcal{I}$.

Definition 2.8 — Exterior differential system generated by an EDS Let $\mathcal{M}$ be a m-dimensional manifold and I an exterior differential system on $\mathcal{M}$. The exterior differential ideal generated by I is the smallest exterior differential ideal containing I.

Examples 2.9 — Exterior differential ideal generated by an EDS on $\mathbb{R}^3$-Continued.

1. **Pfaffian form:** The exterior differential ideal generated by I_1 is:

$$\tilde{\mathcal{I}}_1 = \{\omega\}_{\text{diff}} = \{\alpha \wedge \omega + \beta \wedge \mathrm{d}\omega | \alpha, \beta \in \Gamma(\wedge \mathrm{T}^*(\mathbb{R}^3))\} \tag{2.3}$$

2. **"Non-Pfaffian" system:** The exterior differential ideal generated by I_2 is:

$$\tilde{\mathcal{I}}_2 = \{\omega^1, \omega^2\}_{\text{diff}} = \{\alpha \wedge \omega^1 + \beta \wedge \mathrm{d}\omega^1 + \gamma \wedge \omega^2 + \theta \wedge \mathrm{d}\omega^2 | \alpha, \beta, \gamma, \theta \in \Gamma(\wedge \mathrm{T}^*(\mathbb{R}^3))\} = \{\omega^1, \omega^2\}_{\text{alg}}$$

because both ω^1 and ω^2 are closed.

Definition 2.10 — Closed EDS An exterior differential system $\mathrm{I} \subset \Gamma(\wedge \mathrm{T}^*\mathcal{M})$ is closed if the exterior differentiation of all its differential forms belongs to the exterior ideal generated by I.

Proposition-Definition 2.11 — Closed EDS An exterior differential system $\mathrm{I} \subset \Gamma(\wedge \mathrm{T}^*\mathcal{M})$ is closed if and only if the exterior differential ideal generated by I is equal to the exterior ideal generated by I. In particular, $\mathrm{I} \cup \mathrm{dI}$ is closed.

Examples 2.12 — Closed EDS.

1. The EDS I_2 is closed because the differential of its forms $\mathrm{d}x - \mathrm{d}y$ and $\mathrm{d}z \wedge \mathrm{d}y$ vanishes, and, as shown by example 2.9, the exterior ideal generated by I_2 is equal to the exterior differential ideal generated by I_2.

2. Let $\mathrm{I}_3 = \{\mathrm{d}z - x\mathrm{d}y\}$ be an EDS on $\mathbb{R}^3$. Then I_3 is not closed because the differential of $\mathrm{d}z - x\mathrm{d}y$ is the differential 2-form $-\mathrm{d}x \wedge \mathrm{d}y$ which can not be expressed as the wedge product of the form $\mathrm{d}z - x\mathrm{d}y$ by another differential form on $\mathbb{R}^3$.

Definition 2.13 – Integral manifold Let $\mathcal{M}$ be an m-dimensional manifold, $\mathcal{I} \subset \Gamma(\wedge \mathrm{T}^*\mathcal{M})$ be an exterior differential ideal on $\mathcal{M}$, and $\mathcal{N}$ be a submanifold of $\mathcal{M}$. The submanifold $\mathcal{N}$ is an integral manifold of $\mathcal{I}$ if $\iota^*\varphi = 0, \forall \varphi \in \mathcal{I}$, where ι is an embedding $\iota : \mathcal{N} \longrightarrow \mathcal{M}$.

If an integral manifold of maximal degree exists through each point of $\mathcal{M}$, then the exterior differential ideal (or the EDS) is said to be completely integrable. From the definition of an integral manifold, one can see the need to define exterior differential ideals. Indeed, the pull-back of differential forms commutes with both the wedge product and the exterior differential. In mathematical literature and in this thesis, integral manifolds of an exterior differential system mean integral manifolds of the exterior differential ideal generated by that EDS. The following theorem gives a necessary and sufficient condition for the existence of integral manifolds for Pfaffian systems.

Theorem 2.14 – Frobenius Let $\mathrm{I} = \{\omega^1, \ldots, \omega^r\}$ be a Pfaffian system on an m-dimensional manifold $\mathcal{M}$. Then a necessary and sufficient condition for I to be completely integrable is:

$$\mathrm{d}\omega^i \wedge \omega^1 \wedge \cdots \wedge \omega^r = 0 \quad \text{for all } i = 1, \ldots, r. \tag{2.4}$$

Example 2.15 – Pfaffian equation-Continued. The necessary and sufficient condition for the existence of integral surfaces for the Pfaffian equation $\mathrm{I}_1 = \{a\mathrm{d}x + b\mathrm{d}y + c\mathrm{d}z\}$ is:

$$c\Big(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}\Big) + b\Big(\frac{\partial c}{\partial x} - \frac{\partial a}{\partial z}\Big) + a\Big(\frac{\partial c}{\partial y} - \frac{\partial b}{\partial z}\Big) = 0. \tag{2.5}$$

Hence, according to (2.5):

1. $\mathrm{I}_3 = \{\mathrm{d}z - x\mathrm{d}y\}$ is not completely integrable in $\mathbb{R}^3$.

2. $\mathrm{I}_4 = \{x\mathrm{d}x + y\mathrm{d}y + z\mathrm{d}z\}$ is completely integrable in $\mathbb{R}^3 \setminus \{0\}$.

3. $\mathrm{I}_5 = \{z\mathrm{d}x + x\mathrm{d}y + y\mathrm{d}z\}$ is not completely integrable in $\mathbb{R}^3 \setminus \{0\}$.

The well-known Cauchy theorem assures that given a point M, on a manifold $\mathcal{M}$ and a tangent vector field $\vec{\mathrm{X}}_{\mathrm{M}}$ on $\mathcal{M}$, there exists a curve $\gamma :] - \varepsilon, \varepsilon[$ such that $\gamma(0) = \mathrm{M}$ and $\dot{\gamma}(t) = \vec{\mathrm{X}}_{\gamma(t)}$ for all t. The following corollary of Frobenius theorem may be seen as a differential form version of the Cauchy theorem.

Corollary 2.16 – Cauchy's theorem via differential forms Let $\mathcal{M}^m$ be an m-dimensional manifold and Ibe a Pfaffian system generated by $(m-1)$ linearly independent differential 1-forms. Then I is completely integrable.

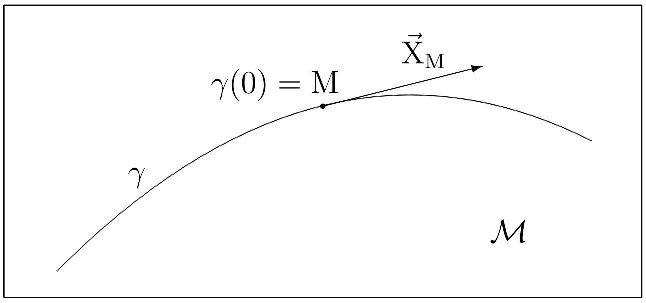

Figure 2.1: Cauchy theorem

Proof. Let $I = \{\omega^1, \dots, \omega^{m-1}\}$, where the differential 1-form ω^i, $i = 1, \dots, m-1$ are linearly independent. One can complete I by a 1-form ω^m to obtain a coframe of $\mathcal{M}$. For $i = 1, \dots, m-1$, the exterior differential of ω^i is expressed in the coframe as follows: $d\omega^i = C^i_{jk}\omega^j \wedge \omega^k$. Therefore, the Frobenius condition is always satisfied since $d\omega^i \wedge \omega^1 \wedge \cdots \wedge \omega^{m-1} = 0$. □

Example 2.17 Let $I_6 = \{x dy - y dx\}$ be an EDS in $\mathbb{R}^2 \setminus \{0\}$. Then according to Proposition 2.16, the EDS I is completely integrable. And more generally, any non-vanishing differential form on a surface is completely integrable.

2.2 Cartan–Kähler theory

If an EDS contains differential 1-forms and functions, we can still apply the Frobenius theorem to the submanifold defined by the vanishing of these functions (except on the possible singularities). However, if the EDS contains differential forms of a degree greater than 1, the Frobenius theorem is no longer helpful, as is often the case for EDSs arising from geometric problems. For instance, let Ω be a closed differential 2-form on an $2m$-dimensional manifold $\mathcal{M}$ such that $\Omega^m \neq 0$. The pair $(\mathcal{M}^{2m}, \Omega)$ is called a *symplectic manifold.* The integral m-dimensional manifolds of $\{\Omega\}$, if they exist, are called Lagrangian manifolds. Thus finding Lagrangian manifolds for a given symplectic manifold is equivalent to looking for integral manifolds of a differential 2-form. Besides the Frobenius theorem, there are standard differential techniques of ordinary differential equations that allow a complete (local) description of integral manifolds of a exterior differential system, like the Pfaff-Darboux and Goursat theorems. The following theory represents a general method for finding and constructing integral manifolds for any exterior differential system.

2.2.1 Integral elements of an EDS and their extensions

Definition 2.18 – Integral element Let $\mathcal{I}$ be an exterior differential ideal on an m-dimensional manifold $\mathcal{M}$. An integral element of $\mathcal{I}$ at a point $M \in \mathcal{M}$ is a linear subspace E of $T_M\mathcal{M}$ such that $\varphi_E = 0$ for all $\varphi \in \mathcal{I}$, where φ_E means the evaluation of φ on any basis of E.

The set of p-dimensional integral elements of $\mathcal{I}$ is denoted by $\mathcal{V}_p(\mathcal{I})$.

Proposition 2.19 – Subspace of an integral element If E is an integral element of the exterior differential ideal $\mathcal{I}$ on $\mathcal{M}$, then every vector subspace of E is also an integral element of $\mathcal{I}$.

Proof. Let W_1 be a vector subspace of an n-integral element E of $\mathcal{I}$ such that W_1 is not an integral element of $\mathcal{I}$. Then, there exists a differential form $\varphi \in \mathcal{I}$, such that $\varphi_{W_1} \neq 0$. Let W_2 be a vector subspace of E such that $E = W_1 \oplus W_2$. Then $\varphi \wedge \psi$, where we choose ψ such that $\psi_{W_1} = 0$ and $\psi_{W_2} \neq 0$, and the degree of φ is $\dim W_1$, belongs to $\mathcal{I}$ and does not annihilate E, contradicting the assumption that E is an integral element of $\mathcal{I}$. □

Proposition 2.20 – Integral elements space of an EDI Let $\mathcal{I}$ be an exterior differential ideal on an m-dimensional manifold. Then $\mathcal{V}_p(\mathcal{I}) = \{E \in G_p(T\mathcal{M}) | \varphi_E = 0 \text{ for all } \varphi \in \mathcal{I}_p\}$

Proof. The containment "$\subset$" is clear by definition. In order to prove the containment "$\supset$", it suffices to show that if $\varphi_E = 0$ for all $\varphi \in \mathcal{I}_p$, then $\varphi_E = 0$ for all $\varphi \in \mathcal{I}$. □

Example 2.21 – Integral element of an EDS.

1. $E_1 = \text{span}\{\partial/\partial x + \partial/\partial y\}$ is an 1-integral element of $I_2 = \{dx - dy, dz \wedge dy\}$ because $(dx - dy)_{E_1}$ vanishes.

2. $\text{span}\{\partial/\partial x - \partial/\partial z, \partial/\partial y - \partial/\partial z\}$ is a 2-integral element of $I_5 = \{dx + dy + dz\}$ in $\mathbb{R}^3$. One can check that any subspace of E_2 is also an integral element of I_5.

Definition 2.22 – Polar space Let E be an integral element of an exterior differential ideal $\mathcal{I}$ on $\mathcal{M}$. Let $\{e_1, e_2, \ldots, e_p\}$ be a basis of $E \subset T_M\mathcal{M}$. The polar space of E, denoted by H(E), is the vector space defined as follows:

$$H(E) = \{v \in T_M\mathcal{M} \,|\, \varphi(v, e_1, e_2, \ldots, e_p) = 0 \text{ for all } \varphi \in \mathcal{I}_{p+1}\}. \tag{2.6}$$

Let us notice that the integral element E is a subset of its polar space. This is due to the fact that a differential form is alternate. The polar space H(E) plays an important role in the EDS theory as shown in the following proposition.

Proposition 2.23 – Extending and integral element Let $E \in \mathcal{V}_p(\mathcal{I})$ be a p-dimensional integral element of $\mathcal{I}$. A $(p+1)$-dimensional vector space $E^+ \subset T_M\mathcal{M}$ which contains E is an integral element of $\mathcal{I}$ if and only if $E^+ \subset H(E)$.

Proof. Suppose that $E^+ = E \oplus \mathbb{R}v$, and let $(e_1, \ldots, e_p)$ be a basis of a p-integral element E of $\mathcal{I}$. The $(p+1)$-subspace is a $(p+1)$-integral element of $\mathcal{I}$ if $\varphi_{E^+} = 0$ for all $\varphi \in \mathcal{I}_{p+1}$. By definition, E^+ is a $(p+1)$-integral element of $\mathcal{I}$ if v belongs to the polar space of E. □

In order to determine if a given p-integral element of an exterior differential system $\mathcal{I}$ is contained in a $(p+1)$-integral element, let us introduce the following function:

Definition 2.24 – Extension rank Let $\mathcal{I}$ be an exterior differential system on an m-dimensional manifold $\mathcal{M}$. Let $r : \mathcal{V}_p(\mathcal{I}) \longrightarrow \mathbb{Z}$ be a function on the integral element of $\mathcal{I}$ with values in integer numbers that associates $E \in \mathcal{V}_p(\mathcal{I})$ with the integer $r(E) = \dim H(E) - (p+1)$.

Remark 2.25 – Extension rank. The extension rank of an integral element of an exterior differential system is always greater or equal to -1. If the extension rank of an integral element E is equal to -1, then $\dim H(E) = \dim E$ so that $H(E) = E$ and consequently, there is no hope of extending the integral element E.

Example 2.26 – Polar space and Extension rank-continued. The EDSs from the previous examples do not contain functions. Consequently, any point is a 0-integral element. Let us then consider M a point in $\mathcal{M}$.

1. **The EDS $\mathbf{I_2}$:** The polar space of $E_0 = M$ is : $H(E_0) = \{\xi \in T_M\mathbb{R}^3 | (dx - dy)(\xi) = 0\}$. Hence, $H(E_0) = \text{span}\{\partial/\partial x + \partial/\partial y, \partial/\partial z\}$. The extension rank of E_0 is $r(E_0) = \dim H(E_0) - 1 = 2 - 1 = 1$. Therefore, there exist 1-integral elements of I_2. Consider then a vector space $E_1 = \text{span}\{\alpha\partial/\partial x + \alpha\partial/\partial y + \beta\partial/\partial z\}$, where α and β are real numbers not simultaneously zero. The polar space of E_1 is then: $H(E_1) = \{\xi \in T_M\mathbb{R}^3 | (dx - dy)(\xi) = dz \wedge dy)(\xi, E_1) = 0\}$. The rank of the polar system is 2, and hence the extension rank of E_1 is $r(E_1) = \dim H(E_1) - 2 = 1 - 2 = -1$. Therefore, there are no 2-integral elements of I_2.

2. **The EDS $\mathbf{I_4}$:** The polar space of E_0 is: $H(E_0) = \{\xi \in T_M\mathbb{R}^3 | (dx + dy + dz)(\xi) = 0\}$. Hence, $H(E_0) = \text{span}\{\partial/\partial x - \partial/\partial z, \partial/\partial y - \partial/\partial z\}$. The extension rank of E_0 is $r(E_0) = \dim H(E_0) - 1 = 2 - 1 = 1$. Therefore, there exist 1-integral elements of I_5. Let us take $E_1 = \text{span}\{\partial/\partial x - \partial/\partial z\}$: since there is no differential 2-form, the polar space of E_1 is $H(E_0)$ and the extension rank is $r(E_1) = 2 - 2 = 0$. There then exists a unique 2-integral element of I_5.

2.2.2 Integral flags, involution and existence theorems

In this subsection, we are interested in determining whether or not an exterior differential system which has a condition of independence admits integral manifolds. Such condition is present, for instance, for exterior differential systems arising from systems of PDEs. It is then compulsory that the defining equations of the integral manifold do not contain relations between the independent variables.

Definition 2.27 – EDS with an independence condition An EDS in an m-dimensional manifold $\mathcal{M}$ with independence condition is a pair (I, Δ) where I is an exterior differential system on $\mathcal{M}$ and Δ is a differential non-vanishing n-form on $\mathcal{M}$.

Exterior ideal and exterior differential ideal with an independence condition are defined like an EDS, i.e., by the assignment of a non-vanishing differential n-form.

Definition 2.28 – Integral elements with an independence condition Let $\mathcal{I}$ be an exterior differential ideal on an m-dimensional manifold $\mathcal{M}$ with an independence condition $\Delta \in \Gamma(\wedge^n T^*\mathcal{M})$, and let $G_n(T\mathcal{M}, \Delta) = \{E \in G_n(T\mathcal{M})/\Delta_E \neq 0\}$ be the Grassmannian manifold of the tangent bundle $T\mathcal{M}$, consisting of the n-dimensional subspaces $T\mathcal{M}$ on which Δ does not vanish. Then the set of integral elements of $(\mathcal{I}, \Delta)$ denoted by $\mathcal{V}_n(\mathcal{I}, \Delta)$ is the set of integral elements of $\mathcal{I}$ on which Δ does not vanish, i.e., $\mathcal{V}_n(\mathcal{I}, \Delta) = \mathcal{V}_n(\mathcal{I}) \cap G_n(TM, \Delta)$.

Consequently, solutions to (I, Δ) are integral manifolds of I on which Δ does not vanish.

Definition 2.29 – Kähler ordinary integral element An n-integral element E of an exterior differential ideal is said to be Kähler ordinary if there exists a differential n-form Δ such that

$\Delta_E \neq 0$ with with the property that E is an ordinary zero of the set of functions $\mathcal{F}_\Delta = \{\varphi_\Delta | \varphi \in \mathcal{I}^n\}$.

Definition 2.30 – Kähler regular integral element A Kähler ordinary n-integral element E of an exterior differential ideal is said to be Kähler regular if the extension function r is locally constant in the neighborhood of E in $G_n(TM, \Delta)$.

Definition 2.31 – Integral flag An integral flag of an exterior differential ideal $\mathcal{I}$ in $M \in \mathcal{M}$ of length n is a sequence $(0)_M \subset E_1 \subset E_2 \subset \cdots \subset E_n \subset T_M\mathcal{M}$ of integral elements E_k of $\mathcal{I}$.

Definition 2.32 – Ordinary and regular integral element An integral element $E \in \mathcal{V}(\mathcal{I})$ is ordinary if its base point $z \in \mathcal{M}$ is an ordinary 0-integral element of $\mathcal{I}$ and if there exists an integral flag $(0)_z \subset E_1 \subset E_2 \subset \cdots \subset E_n = E \subset T_z\mathcal{M}$ where the E_k, $k = 1, \ldots, n-1$ are Kähler regular integral elements. Moreover, if E is Kähler regular, then E is said to be regular.

The following results are intended to check the involution of an EDS, and to show and to construct integral manifolds. The proofs may be found in [Car71, Käh34, BCG+91, IL03]

Theorem 2.33 – Cartan's test Let $\mathcal{I} \subset \Gamma(\wedge^* T^* \mathcal{M})$ be an exterior ideal which does not contain 0-forms (functions on $\mathcal{M}$). Let $(0)_M \subset E_1 \subset E_2 \subset \cdots \subset E_n \subset T_M\mathcal{M}$ be an integral flag of $\mathcal{I}$. For any $k < n$, denote by C_k the codimension of the polar space $H(E_k)$ in $T_M\mathcal{M}$. Then $\mathcal{V}_n(\mathcal{I}) \subset G_n(T\mathcal{M})$ is at least of codimension $C_0 + C_1 + \cdots + C_{n-1}$ at E_n. Moreover, E_n is an ordinary integral flag if and only if E_n has a neighborhood $\mathcal{O}$ in $G_n(T\mathcal{M})$ such that $\mathcal{V}_n(\mathcal{I}) \cap \mathcal{O}$ is a manifold of codimension $C_0 + C_1 + \cdots + C_{n-1}$ in $\mathcal{O}$.

In order to be able to use the Cartan test, the following technical result provides an effective way of computing the characters C_k which are associated with an integral flag.

Proposition 2.34 – A way to compute Cartan characters At a point $M \in \mathcal{M}$, let E be an n-dimensional integral element of an exterior ideal $\mathcal{I}$ which does not contain differential 0-forms. Let $\omega_1, \omega_2, \ldots, \omega_n, \pi_1, \pi_2, \ldots, \pi_s$, where $s = \dim \mathcal{M} - n$, be a coframe in an open neighborhood of $M \in M$ such that $E = \{v \in T_M\mathcal{M} \,|\, \pi_a(v) = 0 \text{ for all } a = 1, \ldots, s\}$. For all $p \leqslant n$, we define $E_p = \{v \in E \,|\, \omega_k(v) = 0 \text{ for all } k > p\}$. Let $\{\varphi_1, \varphi_2, \ldots, \varphi_r\}$ be the set of differential forms which generate the exterior ideal $\mathcal{I}$, where φ_ρ is of degree $(d_\rho + 1)$. Then for all ρ, there exists an expansion

$$\varphi_\rho = \sum_{|J| = d_\rho} \pi_\rho^J \wedge \omega_J + \tilde{\varphi}_\rho \tag{2.7}$$

where the 1-forms π_ρ^J are linear combinations of the forms π and the terms $\tilde{\varphi}_\rho$ are either of degree 2 or more on π, or vanish at z. Moreover, the polar space of E_p is

$$H(E_p) = \{v \in T_M\mathcal{M} | \pi_\rho^J(v) = 0 \text{ for all } \rho \text{ and } \sup J \leqslant p\}. \tag{2.8}$$

In particular, the Cartan characters C_p of the integral flag $(0)_z \subset E_1 \subset E_2 \subset \cdots \subset E_n$ correspond to the number of linear independent forms $\{\pi_\rho^J|_z$ such that $\sup J \leqslant p\}$.

For a differential equation of one variable, the Cauchy problem is well-posed when the initial data is specified:

$$\frac{df}{dt} = F(t, f), \qquad f = y_0 \quad \text{for} \quad t = t_0. \tag{2.9}$$

The Cauchy problem is generalized with several variables and it is, in general, not well-posed. The following theorem provides an answer to the Cauchy problem in higher dimensions:

Theorem 2.35 — Cauchy–Kowalevskaya Let y be a coordinate on $\mathbb{R}$, let $x = (x^i) =$ be coordinates on $\mathbb{R}^n$, let $z = (z^a)$ be coordinates on $\mathbb{R}^s$, and let p_i^a be coordinates on $\mathbb{R}^{ns}$. Let $\mathcal{D} \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^s \times \mathbb{R}^{ns}$ be an open domain, and let $\mathrm{G} : \mathcal{D} \longrightarrow \mathbb{R}^s$ be a real analytic mapping. Let $\mathcal{D}_0 \subset \mathbb{R}^n$ be an open domain and let $f : \mathcal{D}_0 \longrightarrow \mathbb{R}^s$ be a real analytic mapping so that the "1-graph"

$$\Gamma_f = \{(x, y, f(x), \mathrm{D}f(x)) | x \in \mathcal{D}_0\} \tag{2.10}$$

lies in $\mathcal{D}$ for some constante y_0, where $\mathrm{D}f(x) \in \mathbb{R}^{ns}$ is the Jacobian of f described by the condition that $p_i^a(\mathrm{D}f(x)) = \partial f^a/\partial x^i$. Then, there exists an open neighborhood $\mathcal{D}_1 \subset \mathcal{D}_0 \times \mathbb{R}$ of $\mathcal{D}_0 \times \{y_0\}$ and a real analytic mapping $\mathrm{F} : \mathcal{D}_1 \longrightarrow \mathbb{R}^s$ which satsifies the PDE with initial condition

$$\begin{aligned} \partial \mathrm{F}/\partial y &= \mathrm{G}(x, y, \mathrm{F}, \partial \mathrm{F}/\partial x) \\ \mathrm{F}(x, y_0) &= f(x) \quad \text{for all } x \in \mathcal{D}_0. \end{aligned} \tag{2.11}$$

Moreover, F is unique in the sense that any other real-analytic solution to (2.11) agrees with F in some neighborhood of $\mathcal{D}_0 \times \{y_0\}$.

The Cauchy–Kovalevskaya theorem is a local existence theorem. It only asserts that a solution exists in a neighborhood of a point and not in the entire space. As soon as one either considers the global solutions or relaxes the assumption of analyticity, this is no longer the case for the existence and/or the uniqueness. As in [BCG+91], we presented the statement for only one derivative. One can allow higher order derivatives by introducing new variables.

Up until now, we expounded upon two cases where we can assure the existence of integral manifolds: for Pfaffian systems with the Frobenius theorem and for systems of partial differential equation that are in the Cauchy–Kovaleskaya form. The following theorem is of great importance because it generalizes both.

Theorem 2.36 — Cartan–Kähler Let $\mathcal{I} \subset \Gamma(\wedge^* \mathrm{T}^* \mathcal{M})$ be a real analytic exterior differential ideal which does not contain functions. Let $\mathcal{X} \subset \mathcal{M}$ be a p-dimensional connected real analytic Kähler-regular integral manifold of $\mathcal{I}$. Suppose that the extension rank is constant around M, and denote its value by $r = r(\mathcal{X})$, and we assume that $r \geqslant 0$. Let $\mathcal{Z} \subset \mathcal{M}$ be a real analytic submanifold of $\mathcal{M}$ of codimension r which contains $\mathcal{X}$ and such that $\mathrm{T_M}\mathcal{Z}$ and $\mathrm{H}(\mathrm{T_M}\mathcal{X})$ are transversal in $\mathrm{T_M}\mathcal{M}$ for all $\mathrm{M} \in \mathcal{X} \subset \mathcal{M}$. There exists then a $(p+1)$-dimensional connected real analytic integral manifold $\mathcal{Y}$ of $\mathcal{I}$, such that $\mathcal{X} \subset \mathcal{Y} \subset \mathcal{Z}$. Moreover, $\mathcal{Y}$ is unique in the sense that another integral manifold of $\mathcal{I}$ having the stated properties coincides with $\mathcal{Y}$ on an open neighborhood of $\mathcal{X}$.

The analycity condition of the exterior differential ideal is crucial because of the requirements in the Cauchy–Kovalevskaya theorem used in the Cartan–Kähler theorem's proof. It has an important corollary. Actually, in the application, this corollary is more often used than the theorem and is sometimes called *the Cartan–Kähler theorem* in mathematical literature.

Corollary 2.37 — Cartan–Kähler Let $\mathcal{I}$ be an analytic exterior differential ideal on a manifold $\mathcal{M}$. If $\mathrm{E} \subset \mathrm{T_M}\mathcal{M}$ is an ordinary integral element of $\mathcal{I}$, there exists an integral manifold of $\mathcal{I}$ passing through z and having E as a tangent space at point M.

Chapter 3

Some surface embedding results

This chapter is dedicated to presenting some embedding results concerning surfaces. This chapter will also present examples of application in differential geometry from the previous two chapters. The embedding results are as follows: Lagrangian embeddings, isometric embeddings, and isometric Lagrangian embeddings of real analytic Riemannian surfaces. The first and second results admit generalizations in higher dimensions and are expounded upon in the subappendix of this chapter.

3.1 Lagrangian surfaces

Symplectic geometry is quite vast and has many interesting results as well as numerous connections to other great theories. However, we will merely introduce and define only the basic objects needed in this chapter. The reader may refer to [Ber01, AdS03] for further investigation.

Definition 3.1 – Symplectic manifold A symplectic manifold is a pair $(\mathcal{M}, \omega)$ where $\mathcal{M}$ is a differentiable manifold and ω is a closed nondegenerate differential 2-form on $\mathcal{M}$. Such a form ω is called a symplectic form.

It results from the assumption on ω that each tangent space $\mathrm{T_M}\mathcal{M}$ at a point M is endowed with a symplectic structure, ie., $(\mathrm{T_M}\mathcal{M}, \omega_\mathrm{M})$ is a symplectic vector space. Hence, the dimension of the manifold must be even. Notice that symplectic vector spaces are symplectic manifolds.

Examples 3.2 – Symplectic manifolds. The following list of examples is not exhaustive.

1. **Cotangent bundle** Let $\mathcal{M}$ be a differentiable manifold and consider its cotangent bundle $(\mathrm{T}^*\mathcal{M}, \pi, \mathcal{M})$. There exists a canonical differential 1-form α on $\mathrm{T}^*\mathcal{M}$, called the Liouville form, defined by $\alpha_{(\mathrm{M},\varphi_\mathrm{M})}(\mathrm{X}) := \varphi_\mathrm{M}(\pi_{*,(\mathrm{M},\varphi_\mathrm{M})}\mathrm{X})$, where $(\mathrm{M}, \varphi_\mathrm{M}) \in \mathrm{T}^*\mathcal{M}$, $\mathrm{X} \in \mathrm{T}_{(\mathrm{M},\varphi_\mathrm{M})}(\mathrm{T}^*\mathcal{M})$ and one can easily check that $\omega = \mathrm{d}\alpha$ is closed and nondegenerate.

2. **Orientable surfaces** Any differential 2-form is closed on a surface. The nondegeneracy condition means that the 2-form does not vanish anywhere, i.e., the 2-form is a volume form. Therefore, all orientable surfaces may be considered as symplectic manifolds. In particular, consider the unit sphere in $\mathbb{R}^3$, whose tangent space at a point M is the orthogonal to the unit vector $\overrightarrow{\mathrm{OM}}$. Then the differential 2-form $\omega_\mathrm{M}(\mathrm{X}, \mathrm{Y}) := \overrightarrow{\mathrm{OM}}.(\mathrm{X}, \mathrm{Y}) = \det(\overrightarrow{\mathrm{OM}}, \mathrm{X}, \mathrm{Y})$ is nondegenerate and thus a symplectic form.

3. **Kähler manifolds** are symplectic.

A fundamental theorem in symplectic geometry is *Darboux*'s theorem.

Theorem 3.3 – Darboux Let $(\mathcal{M}, \omega)$ be a $2m$-dimensional symplectic manifold. For all M in $\mathcal{M}$, there exists a coordinate system $(x_1, \ldots, x_{2m})$ at the point M where

$$\omega_{\mathrm{M}} = \sum_{i=1}^{m} \mathrm{d}x^i \wedge \mathrm{d}x^{i+m}. \tag{3.1}$$

Darboux's theorem affirms that locally, all symplectic forms are isomorphic to each other (only for an even dimension), and hence there is only one model of symplectic manifold for any given even dimension. This rigidity result constitutes the major difference between Riemannian and symplectic geometry and indicates that symplectic geometry is essentially a global theory.

Definition 3.4 – Lagrangian immersion Let $(\mathcal{M}, \omega)$ be a $2m$-dimensional symplectic manifold and $\mathcal{N}$ a submanifold of $\mathcal{M}$. An immersion $f : \mathcal{N} \longrightarrow \mathcal{M}$ is Lagrangian if $f^*(\omega) = 0$ and $\dim\mathcal{N} = m$.

From the above definition, it is then natural to define Lagragian manifolds as follows:

Definition 3.5 – Lagrangian manifold Let $(\mathcal{M}, \omega)$ be a $2m$-dimensional symplectic manifold. A Lagrangian manifold of $(\mathcal{M}, \omega)$ is an m-integral manifold of $\{\omega\}$.

In the following proposition, the existence of Lagrangian surface in the 4-dimensional symplectic space is shown. In higher dimensions cf. to the subappendix of this chapter.

Proposition 3.6 – Lagrangian manifolds in $\mathbb{R}^4$ There exist Lagrangian surfaces of the 4-dimensional symplectic space $(\mathbb{R}^4, \omega)$.

It is straightforward to notice that on $\mathbb{R}^4$ equipped with the symplectic form $\mathrm{d}x^1 \wedge \mathrm{d}x^3 + \mathrm{d}x^2 \wedge \mathrm{d}x^4$, the plane defined by $x^3 = x^4 = 0$ having $\mathrm{span}\{\partial/\partial x^1, \partial/\partial x^2\}$ for a tangent space, is a Lagrangian plane that does not vanish on $\mathrm{d}x^1 \wedge \mathrm{d}x^2$. However, can one expect to find Lagrangian surfaces that are not flat? The solution is well-known and are described as

$$\{x^1, x^2, \frac{\partial \mathrm{S}}{\partial x^1}(x), \frac{\partial \mathrm{S}}{\partial x^2}(x)\} \tag{3.2}$$

The following proof, which includes the case of a plane, is another way of establish this result using the Cartan–Kähler theory.

Proof. Let $(\mathbb{R}^4, \mathrm{d}x^1 \wedge \mathrm{d}x^3 + \mathrm{d}x^2 \wedge \mathrm{d}x^4)$ be a symplectic space. Let us look for Lagrangian surfaces, i.e., integral surfaces for Ω. Since $\Omega = \mathrm{d}x^1 \wedge \mathrm{d}x^3 + \mathrm{d}x^2 \wedge \mathrm{d}x^4$ is closed, the exterior ideal $\mathcal{I}$ generated by Ω is closed. The EDI $\mathcal{I}$ contains neither function nor differential 1-forms, thus any point M of $\mathbb{R}^4$ is an integral point and any tangent vector on $\mathrm{T_M}\mathbb{R}^4 \simeq \mathbb{R}^4$ is an integral 1-element. Denote by E_0 a given point M on $\mathbb{R}^4$. Notice that $r_0 = 3$ and consider E_1 such that $\mathrm{d}x^1_{\mathrm{E}_1} \neq 0$. Then $\mathrm{E}_1 = \mathrm{span}\{\partial/\partial x^1 + \alpha^3_1 \partial/\partial x^3 + \alpha^4_1 \partial/\partial x^4\}$ to be an integral 1-element of $\mathcal{I}$, where α^3_1 and α^4_1 are functions on $\mathbb{R}^4$. The associated polar space is $\mathrm{H}(\mathrm{E}_1) = \{\xi \in \mathrm{T_M}\mathbb{R}^4 | \Omega(\mathrm{X}, \mathrm{E}_1) = 0\}$. Then $\mathrm{H}(\mathrm{E}_1)$ is defined by the equation $\alpha^3_1\xi^1 - \xi^3 - \alpha^4_1\xi^2 = 0$, where ξ^i are the components of the tangent vector ξ on the frame $\{\partial/\partial x^i\}$. Therefore, $\mathrm{C}_1 = 1$ and the extension rank is $r_1 = 3 - 2 = 1$, and hence there exists a 2-integral element of $\{\omega\}$. Consider $\mathrm{E}_2 = \mathrm{span}\{\partial/\partial x^1 + \alpha^3_1\partial/\partial x^3 + \alpha^4_1\partial/\partial x^4, \partial/\partial x^2 + \alpha^4_1\partial/\partial x^3 + \alpha^4_2\partial/\partial x^4\}$ to be an integral 2-element of $\mathcal{I}$. Since the extension ranks r_0, r_1 and r_2 are constant on a neighborhood of M, the flag $\mathrm{M} \subset \mathrm{E}_1 \subset \mathrm{E}_2$ is regular, but let us check the involution by the Cartan test. On a point E of the Grassmannian $\mathrm{G}_2(\mathrm{T_M}\mathbb{R}^4, \mathrm{d}x^1 \wedge \mathrm{d}x^2)$, there exists a unique basis: $(\mathfrak{X}^1(\mathrm{E}), \mathfrak{X}^2(\mathrm{E}))$, where $\mathfrak{X}^i = \partial/\partial x^i + \mathrm{P}^3_i(\mathrm{E})\partial/\partial x^3 + \mathrm{P}^4_i(\mathrm{E})\partial/\partial x^4$. Hence $\mathrm{P}^3_1, \mathrm{P}^3_2, \mathrm{P}^4_1, \mathrm{P}^4_2$ is a set of coordinates on $\mathrm{G}_2(\mathrm{T_M}\mathbb{R}^4, \mathrm{d}x^1 \wedge \mathrm{d}x^2)$, the vanishing of the pull-back of the symplectic form on the Grassmannian reads $\mathrm{P}_2^3 - \mathrm{P}_1^4 = 0$, whose differential is obviously linearly independent, and hence $\mathrm{codim}(\mathcal{V}_2(\mathcal{I}, \mathrm{d}x^1 \wedge \mathrm{d}x^2) = 1$. The exterior ideal $\{\omega\}_{\mathrm{alg}}$ passes the Cartan test because $\mathrm{C}_0 + \mathrm{C}_1 = \mathrm{codim}\mathcal{V}_2(\mathcal{I}, \mathrm{d}x^1 \wedge \mathrm{d}x^2)$, and hence the EDS is in involution. Furthermore, the Cartan–Kähler theorem demonstrates that there are integral manifolds of the symplectic form. By construction, the Lagrangian surface satisfies the independence condition $\mathrm{d}x^1 \wedge \mathrm{d}x^2$. □

3.2 ISOMETRIC EMBEDDING OF SURFACES

The following proposition is a special case of the Cartan–Janet theorem, a proof of which is later given in the subappendix of this chapter. Although it is included in the Cartan–Janet proof, the following proposition will be expounded upon not only for the further understanding of the reader but also for further use in the embedding results and in the following chapters.

Proposition 3.7 – Isometric embedding of surfaces Every real analytic Riemannian surface $(\mathcal{M}^2, g)$ can be, locally, isometrically embedded in a three dimensional Euclidean space. Moreover, the local isometric embedding depends on two functions of one variable.

Proof. Let $(\mathrm{E}_1, \mathrm{E}_2)$ be a g-orthonormal moving frame in the neighborhood of a point M of $\mathcal{M}^2$ and denote the associated moving coframe by (η^1, η^2). Let $\mathcal{F}_2(\mathbb{R}^3) \simeq \mathbb{R}^3 \times \mathrm{SO}(3)$ be the vector bundle over $\mathbb{R}^3$ consisting of pairs of orthonormal vectors of $\mathbb{R}^3$ defined as follows: $\mathcal{F}_2(\mathbb{R}^3) = \{(\mathrm{N}, e_1, e_2) | \mathrm{N} \in \mathbb{R}^3$ and $(e_1, e_2, e_3 = e_1 \times e_2)$ is a direct orthonormal basis of $\mathbb{R}^3\}$. Thus, we obtain on $\mathcal{F}_2(\mathbb{R}^3)$ the coframe $\{\omega^1, \omega^2, \omega^3, \omega_2^1, \omega_1^3, \omega_2^3\}$ defined as follows:

$$\omega^a = \langle e_a, \mathrm{d}x \rangle \qquad \text{and} \qquad \omega_b^a = \langle e_a, \mathrm{d}e_b \rangle \tag{3.3}$$

Let $\{\eta^1, \eta^2, \omega^1 - \eta^1, \omega^2 - \eta^2, \omega^3, \omega_2^1 - \eta_2^1, \omega_1^3, \omega_2^3\}$ be a moving coframe of $\Sigma = \mathcal{M}^2 \times \mathcal{F}_2(\mathbb{R}^3)$. Let $\mathcal{I} = \{\omega^1 - \eta^1, \omega^2 - \eta^2, \omega^3, \omega_2^1 - \eta_2^1, \omega_1^3 \wedge \eta^1 + \omega_2^3 \wedge \eta^2, \omega_1^3 \wedge \omega_2^3 - \mathcal{K}\eta^2 \wedge \eta^2\}$ be a closed exterior differential ideal, where $\mathcal{K}$ is the Gauss curvature of the surface $(\mathcal{M}^2, g)$.

COMPUTING CODIM$\mathbf{V_2(\mathcal{I}, \eta^1 \wedge \eta^2)}$:

In order to facilitate the computation of the codimension of 2-integral elements that do not vanish on $\eta^1 \wedge \eta^2$, we change the notation: $\{\eta^1, \eta^2, \varpi^1, \varpi^2, \varpi^3, \varpi^4, \varpi^5, \varpi^6\}$ now denotes $\{\eta^1, \eta^2, \omega^1 - \eta^1, \omega^2 - \eta^2, \omega^3, \omega_2^1 - \eta_2^1, \omega_1^3, \omega_2^3\}$. The exterior differential ideal $\mathcal{I}$ is newly written as follows:

$$\mathcal{I} = \{\varpi^1, \varpi^2, \varpi^3, \varpi^4, \varpi^5 \wedge \eta^1 + \varpi^6 \wedge \eta^2, \varpi^5 \wedge \varpi^6 - \mathcal{K}\eta^1 \wedge \eta^2\}. \tag{3.4}$$

Let $\{\mathfrak{X}_1(\mathrm{E}), \mathfrak{X}_2(\mathrm{E})\}$ be a basis of $\mathrm{G}_2(\mathrm{T}\Sigma, \eta^1 \wedge \eta^2)$, the space of 2-planes of $\mathrm{T}\Sigma$ that do not annihilate the differential 2-form $\eta^1 \wedge \eta^2$. The vectors $\mathfrak{X}_i(\mathrm{E})$ are defined as shown here:

$$\mathfrak{X}_1(\mathrm{E}) = \mathrm{X}_1 + \mathrm{P}_1^1(\mathrm{E})\mathrm{Y}_1 + \mathrm{P}_1^2(\mathrm{E})\mathrm{Y}_2 + \cdots + \mathrm{P}_1^6(\mathrm{E})\mathrm{Y}_6 \tag{3.5}$$

$$\mathfrak{X}_2(\mathrm{E}) = \mathrm{X}_2 + \mathrm{P}_2^1(\mathrm{E})\mathrm{Y}_1 + \mathrm{P}_2^2(\mathrm{E})\mathrm{Y}_2 + \cdots + \mathrm{P}_2^6(\mathrm{E})\mathrm{Y}_6 \tag{3.6}$$

where the tangent vector family $\{\mathrm{X}_1\}$ is the dual basis of $\{\eta^i\}$, and the tangent vector family $\{\mathrm{Y}_a\}$ is the dual of $\{\varpi^a\}$. Now we can express the differential forms that generate $\mathcal{I}$ on $\mathrm{G}_2(\mathrm{T}\Sigma, \eta^1 \wedge \eta^2)$. These forms are denoted by the same symbols with an index E and we evaluate these forms on the basis $\mathfrak{X}_i(\mathrm{E})$. We have $\varpi^i(\mathfrak{X}_i(\mathrm{E})) = \mathrm{P}^i_j(\mathrm{E})$. So, $\varpi^i_{\mathrm{E}} = \mathrm{P}^i_j\Pi^j$, where $i = 1, \ldots, 4$ and $j = 1, 2.$, and $\{\Pi^i\}$ is the dual basis of $\{\mathfrak{X}_i\}$.

$$(\varpi^5 \wedge \eta^1 + \varpi^6 \wedge \eta^2)(\mathrm{X}_1(\mathrm{E}), \mathrm{X}_2(\mathrm{E})) = \mathrm{P}^6_1 - \mathrm{P}^5_2 \tag{3.7}$$

Hence, $\mathrm{d}\varpi^3_{\mathrm{E}} = -(\varpi^5 \wedge \eta^1 + \varpi^6 \wedge \eta^2)_{\mathrm{E}} = -(\mathrm{P}^6_1 - \mathrm{P}^5_1)\Pi^1 \wedge \Pi^2$.

$$(\varpi^5 \wedge \varpi^6 - \mathcal{R}\eta^1 \wedge \eta^2)(\mathrm{X}_1(\mathrm{E}), \mathrm{X}_2(\mathrm{E})) = \mathrm{P}^5_1\mathrm{P}^6_2 - \mathrm{P}^6_1\mathrm{P}^5_2 - \mathcal{K} \tag{3.8}$$

Hence, $\mathrm{d}\varpi^4_{\mathrm{E}} = (\varpi^5 \wedge \varpi^6 - \mathcal{K}\eta^1 \wedge \eta^2)_{\mathrm{E}} = (\mathrm{P}^5_1\mathrm{P}^6_2 - \mathrm{P}^6_1\mathrm{P}^5_2 - \mathcal{K})\Pi^1 \wedge \Pi^2$.

The vanishing of the differential forms $\varpi^1_{\mathrm{E}}, \varpi^2_{\mathrm{E}}, \varpi^3_{\mathrm{E}}, \varpi^4_{\mathrm{E}}, (\varpi^5 \wedge \eta^1 + \varpi^6 \wedge \eta^2)_{\mathrm{E}}$ and $(\varpi^5 \wedge \varpi^6 - \mathcal{R}\eta^\wedge\eta^2)_{\mathrm{E}}$ is equivalent to the system

$$\mathrm{P}^1_1 = \mathrm{P}^1_2 = \mathrm{P}^2_1 = \mathrm{P}^2_2 = \mathrm{P}^3_1 = \mathrm{P}^3_2 = \mathrm{P}^4_1 = \mathrm{P}^4_2 = \mathrm{P}^6_1 - \mathrm{P}^5_2 = \mathrm{P}^5_1\mathrm{P}^6_2 - \mathrm{P}^6_1\mathrm{P}^5_2 - \mathcal{K} = 0. \tag{3.9}$$

These six relations, which are linearly independent, define $\mathrm{V}_2(\mathcal{I}, \eta^1 \wedge \eta^2)$ the space of 2-integral elements of $\mathcal{I}$ that do not vanish on $\eta^1 \wedge \eta^2$. Consequently, the codimension of $\mathrm{V}_2(\mathcal{I}, \eta^1 \wedge \eta^2)$ in $\mathrm{G}_2(\mathrm{T}\Sigma, \eta^1 \wedge \eta^2)$ is 10.

CONSTRUCTING AN ORDINARY 2-FLAG OF $\mathcal{I}$

If we construct an ordinary integral flag of length 2, then the Cartan–Kähler theorem assures the existence and the uniqueness of a 2-integral manifold of $\mathcal{I}$ with the independence condition $\eta^1 \wedge \eta^2$. A tangent vector $\xi \in \mathrm{T}\Sigma$ is expressed as follows:

$$\xi = \xi^1_{\mathcal{M}}\mathrm{X}_1 + \xi^2_{\mathcal{M}}\mathrm{X}_2 + \xi^1\mathrm{Y}_1 + \xi^2\mathrm{Y}_2 + \cdots + \xi^6\mathrm{Y}_6 \tag{3.10}$$

The exterior differential ideal $\mathcal{I}$ does not contain functions (0-forms) . Therefore, every point of Σ is an integral point. Consider $z = \mathrm{E}_0 \in \Sigma$. The polar space of E_0 is:

$$\mathrm{H}(\mathrm{E}_0) = \{\xi \in \mathrm{T}_z\Sigma | \varpi^1(\xi) = \varpi^2(\xi) = \varpi^3(\xi) = \varpi^4(\xi) = 0\}. \tag{3.11}$$

Every tangent vector ξ satisfying $\xi^1 = \xi^2 = \xi^3 = \xi^4 = 0$ belongs to the polar space of E_0. The codimension of $\mathrm{H}(\mathrm{E}_0)$ is $\mathrm{C}_0 = 4$. The extension rank is $r_0 = \mathrm{dimH}(\mathrm{E}_0) - 1 = 4 - 1 = 3$. Therefore, there exist 1-integral elements. For instance, $\mathrm{E}_1 = (z, e_1)$ where $e_1 = \mathrm{X}_1 + \alpha^5_1\mathrm{Y}_5 + \alpha^6_1\mathrm{Y}_6$ with the condition $\alpha^5_1 \neq 0$. The polar space of E_1 is

$$\mathrm{H}(\mathrm{E}_1) = \{\xi \in \mathrm{T}_z\Sigma | \varpi^1(\xi) = \cdots = \varpi^4(\xi) = (\varpi^5 \wedge \eta^1 + \varpi^6 \wedge \eta^2)(\xi, e_1) = (\varpi^5 \wedge \varpi^6 - \mathcal{R}\eta^1 \wedge \eta^2)(\xi, e_1) = 0\}. \tag{3.12}$$

where

$$(\varpi^5 \wedge \eta^1 + \varpi^6\eta^2)(\xi, e_1) = \xi^5 - \alpha^5_1\xi^1_{\mathcal{M}} - \alpha^6_1\xi^2_{\mathcal{M}} = 0 \tag{3.13}$$

and

$$(\varpi^5 \wedge \varpi^6 - \mathcal{R}\eta^1 \wedge \eta^2)(\xi, e_1) = \alpha^6_1\xi^5 - \alpha^5_1\xi^6 + \mathcal{K}\xi^2_{\mathcal{M}} = 0 \tag{3.14}$$

Every tangent vector ξ satisfying $\xi^1 = \xi^2 = \xi^3 = \xi^4 = \xi^4 = (\xi^5 - \alpha^5_1\xi^1_{\mathcal{M}} - \alpha^6_1\xi^2_{\mathcal{M}}) = 0$ and $\alpha^6_1\xi^5 - \alpha^5_1\xi^6 + \mathcal{K}\xi^2_{\mathcal{M}} = 0$ belongs to the polar space of E_1. The codimension of $\mathrm{H}(\mathrm{E}_1)$ is $\mathrm{C}_1 = 6$. The extension rank $r_1 = \mathrm{dimH}(\mathrm{E}_1) - 2 = 2 - 2 = 0$. We can thus conclude that there exists a 2-integral element. For instance, $\mathrm{E}_2 = (z, e_1, e_2)$ where $e_2 = \mathrm{X}_2 + \alpha^5_2\mathrm{Y}_5 + \alpha^6_2\mathrm{Y}_6$ with $\alpha^5_1 = \alpha^6_1$ and $\alpha^6_2 = ((\alpha^6_1)^2 + \mathcal{K})/\alpha^5_1$.

We therefore constructed an integral flag $(0)_z \subset \mathrm{E}_1 \subset \mathrm{E}_2 = \mathrm{E}$. This flag is ordinary since $\mathrm{C}_0 + \mathrm{C}_1 = 4 + 6 = 10 = \mathrm{CodimV}_2(\mathcal{I}, \eta^1 \wedge \eta^2)$, and hence passes the Cartan test. We conclude then that the exterior differential system $\mathcal{I}$ is in involution, and according to the Cartan–Kähler theorem, there exists a local isometric embedding of $(\mathcal{M}^2, g)$. □

Remark 3.8 The coefficients $\alpha_1^5, \alpha_2^5, \alpha_1^6$ and α_2^6 in the above proof represent the coefficients of the second fundamental form and satisfy the Gauss equation $\alpha_1^5\alpha_2^6 - \alpha_2^5\alpha_1^6 = \mathcal{K}$.

3.3 Isometric Lagrangian embedding of surfaces

The two previous results show that a real analytic Riemannian surface can be realized locally as a Lagrangian submanifold of the symplectic space $(\mathbb{R}^4, \omega)$ and also as a submanifold of the Euclidean space $(\mathbb{R}^3, \langle,\rangle_{\mathbb{R}^3})$, and hence in a higher Euclidean space. The question if one can expect to have both naturally follows. A reasonable target space is then $\mathbb{C}^2$ since it is a real 4-dimensional vector space and since its complex structure provides a Euclidean and symplectic structure[1].

Theorem 3.9 – Moore–Morvan Let $(\mathcal{M}^2, g)$ be a real analytic Riemannian manifold of dimension two. If $\mathrm{M} \in \mathcal{M}^2$, then there is an open neighborhood $\mathcal{O}$ of M which possesses an isometric Lagrangian immersion into $\mathbb{C}^2$. Indeed, the local isometric Lagrangian immersions depend upon three functions of a single variable.

The following proof closely resembles to one expounded upon in [MM01] except for the fact that we are using a different complex structure, which is not of major importance. Indeed, the symplectic structure underlying the complex one in [MM01] is of the form $\sum \mathrm{d}x^i \wedge \mathrm{d}x^{i+m}$ while the one in the following proof is of the form $\sum \mathrm{d}x^{2i-1} \wedge \mathrm{d}x^{2i}$.

Proof. Let $(\mathcal{M}^2, g)$ be a real analytic Riemannian surface. Let (η^1, η^2) be an orthonormal moving coframe such that $g = \eta^1 \otimes \eta^1 + \eta^2 \otimes \eta^2$. There is then a unique torsion-free g-compatible connection. This connection is determined by a matrix-valued differential 1-form (η^i_j) and satisfies Cartan's structure equations:

$$\mathrm{d}\eta^i + \eta^i_j \wedge \eta^j = 0 \text{ for } i = 1 \text{ and } 2 \qquad \text{and} \qquad \mathrm{d}\eta^1_2 = \Omega^1_2 = \mathcal{K}\eta^1 \wedge \eta^2 \tag{3.15}$$

where Ω^1_2 and $\mathcal{K}$ are the curvature 2-form of the Levi-Civita connection and the Gauss curvature, respectively. Consider $\mathcal{F}_{\mathbb{C}}(\mathbb{C}^n)$ to be the unitary frame bundle of $\mathbb{C}^n$. An element of $\mathcal{F}_{\mathbb{C}}(\mathbb{C}^n)$ is a pair $\Big(\mathrm{N}, (e_1, \dots e_{2n})\Big)$, where $\mathrm{N} \in \mathbb{C}^n$, J is the standard complex structure of $\mathbb{C}^n$ and $(e_1, \dots e_{2n})$ is a real orthonormal frame such that $\mathrm{J}e_{2p-1} = e_{2p}$ for $p = 1, \dots, n$. The bundle $\mathcal{F}_{\mathbb{C}}(\mathbb{C}^n)$ is diffeomorphic to $\mathbb{C}^n \times \mathrm{U}(n)$. Let us define on $\mathcal{F}_{\mathbb{C}}(\mathbb{C}^n)$ the differential 1-forms ω^λ and ω^λ_μ as follows:

$$\mathrm{d}e_\lambda = \sum_{\mu=1}^{2n} e_\mu \omega^\mu_\lambda \qquad \text{and} \qquad \mathrm{d}z = \sum_{\lambda=1}^{2n} e_\lambda \omega^\lambda. \tag{3.16}$$

These differential 1-forms satisfy Cartan's structure equations:

$$\mathrm{d}\omega^\lambda + \omega^\lambda_\mu \wedge \omega^\mu \quad \text{and} \quad \mathrm{d}\omega^\lambda_\mu + \omega^\lambda_\nu \wedge \omega^\nu_\mu = 0. \tag{3.17}$$

The matrix of differential 1-forms $\omega = (\omega^\lambda_\mu)$ takes values in the Lie algebra $\mathfrak{u}(n)$. This implies not only that (ω^λ_μ) is skew-symmetric, but also that

$$\omega^{2p}_{2k-1} = -\omega^{2p-1}_{2k} \qquad \text{and} \qquad \omega^{2p-1}_{2k-1} = \omega^{2p}_{2k} \quad \text{for } p, k = 1, \dots, n. \tag{3.18}$$

[1] The existence of Lagrangian manifolds in $\mathbb{C}^m$ is expounded in appendix 2 as an application to tableaux and linear Pfaffian system.

Indeed, $\mathrm{J}\mathrm{d}e_{2k-1} = \mathrm{d}e_{2k} := \sum_{p=1}^{n} e_{2p-1}\omega_{2k}^{2p-1} + \sum_{p=1}^{n} e_{2p}\omega_{2k}^{2p}$. By multiplying both sides by $-\mathrm{J}$, we obtain

$$\mathrm{d}e_{2k-1} = -\mathrm{J}\mathrm{J}\mathrm{d}e_{2k-1} = \mathrm{d}e_{2k-1} = -\sum_{p=1}^{n} e_{2p-1}\omega_{2k-1}^{2p-1} + \sum_{p=1}^{n} e_{2p}\omega_{2k-1}^{2p}. \tag{3.19}$$

$\mathcal{F}_{\mathbb{C}}(\mathbb{C}^2)$ is (real) 8-dimensional, because it is diffeomorphic to $\mathbb{C}^2 \times \mathrm{U}(2)$. Since $\omega_1^3 = \omega_2^4$ and $\omega_1^4 = -\omega_2^3$, we conclude that $\{\omega^1, \omega^2, \omega^3, \omega^4, \omega_3^1, \omega_1^2, \omega_3^2, \omega_3^4\}$ is a coframe of $\mathcal{F}_{\mathbb{C}}(\mathbb{C}^2)$. Consider on $\Sigma = \mathcal{M}^2 \times \mathcal{F}_{\mathbb{C}}(\mathbb{C}^2)$ the moving coframe $\{\eta^1, \eta^2, \omega^1 - \eta^1, \omega^3 - \eta^2, \omega^2, \omega^4, \omega_3^1 - \eta_2^1, \omega_1^2, \omega_3^2, \omega_3^4\}$ that is denoted, for simplicity, as $\{\eta^1, \eta^2, \varpi^1, \varpi^2, \varpi^3, \varpi^4, \varpi^5, \varpi^6, \varpi^7, \varpi^8\}$. $\mathcal{M}^2$ is said to be a Lagrangian manifold of $\mathbb{C}^2$ if the complex structure J maps the tangent plane of $\mathcal{M}^2$ into the normal fiber of $\mathcal{M}^2$ in $\mathbb{C}^2$. Another way to express this idea is the vanishing of ω^2 and ω^4 on Σ. Moreover, if $\omega^1 - \eta^1$ and $\omega^3 - \eta^2$ vanish on Σ, then the resulting Lagrangian embedding is isometric. By Cartan's structure equations, we have, modulo the forms $(\omega^1 - \eta^1, \omega^3 - \eta^2, \omega^2, \omega^4)$:

$$\begin{cases} \mathrm{d}(\omega^1 - \eta^1) & \equiv & -(\omega_3^1 - \eta_2^1) \wedge \eta^2 \\ \mathrm{d}(\omega^3 - \eta^2) & \equiv & -(\omega_1^3 - \eta_1^2) \wedge \eta^1 \\ \mathrm{d}\omega^2 & \equiv & -\omega_1^2 \wedge \eta^1 - \omega_3^2 \wedge \eta^2 \\ \mathrm{d}\omega^4 & \equiv & -\omega_1^4 \wedge \eta^1 - \omega_3^4 \wedge \eta^2 \end{cases} \tag{3.20}$$

The first two equations imply that the differential 1-form $(\omega_3^1 - \eta_2^1)$ vanishes. Consequently, its exterior differential must also vanish, and provides a Gauss equation type:

$$\omega_1^2 \wedge \omega_3^2 + \omega_1^4 \wedge \omega_3^4 = \mathcal{K}\eta^1 \wedge \eta^2 \tag{3.21}$$

Consider $\mathcal{I}$ to be the exterior ideal generated by the forms $\{\omega^1 - \eta^1, \omega^3 - \eta^2, \omega^2, \omega^4, \omega_3^1 - \eta_2^1, \mathrm{d}\omega^2, \mathrm{d}\omega^4, \mathrm{d}(\omega_3^1 - \eta_2^1)\}$. This ideal is closed under the exterior differentiation. As in the previous proof, the EDI $\mathcal{I}$ is expressed as follows:

$$\mathcal{I} = \{\varpi^1, \dots, \varpi^5, \varpi^6 \wedge \eta^1 + \varpi^7 \wedge \eta^2, \varpi^7 \wedge \eta^1 + \varpi^8 \wedge \eta^2, \varpi^6 \wedge \varpi^7 + \varpi^7 \wedge \varpi^8 - \mathcal{K}\eta^1 \wedge \eta^2\} \tag{3.22}$$

Proposition 3.10 Every integral manifold of $\mathcal{I}$, for which the differential 2-form $\eta^1 \wedge \eta^2$ does not vanish, is locally the graph of map $f : \mathcal{M}^2 \longrightarrow \mathcal{F}_{\mathbb{C}}(\mathbb{C}^2)$ having the property that $u = \pi_{\mathbb{C}^2} \circ f$ is a Lagrangian isometric embedding.

In order to show the existence of an integral manifold of $\mathcal{I}$ for which the 2-form $\eta^1 \wedge \eta^2$ does not vanish, we constuct an ordinary 2-integral element of $\mathcal{I}$. The Cartan–Kähler theorem assures then the existence and uniqueness of an integral manifold, and hence, by the above proposition, the existence of a Lagrangian isometric embedding.

CONSTRUCTION OF AN ORDINARY 2-FLAG

As in the previous proof, denote the coordinates of a tangent vector by

$$\xi = \xi_{\mathcal{M}}^1 \mathrm{X}_1 + \xi_{\mathcal{M}}^2 \mathrm{X}_2 + \xi^1 \mathrm{Y}_1 + \dots \xi^8 \mathrm{Y}_8. \tag{3.23}$$

The exterior differential ideal $\mathcal{I}$ does not contain 0-forms. Every point of Σ is then an integral point of $\mathcal{I}$. Let $z = \mathrm{E}_0$ be a fixed point of Σ. The polar space $\mathrm{H}(\mathrm{E}_0) = \{\xi \in \mathrm{T}_z\Sigma | \varpi^a(\xi) = 0, \text{ for } a = 1, \dots, 5\}$. Every tangent vector ξ such that $\xi^1 = \xi^2 = \cdots = \xi^5 = 0$ belongs to the polar space of E_0. Therefore, $\mathrm{C}_0 = 5$ and $r_0 = \dim \mathrm{H}(\mathrm{E}_0) - 1 = 5 - 1 = 4$. There then exist 1-integral elements of $\mathcal{I}$. Consider e_1 to be defined as follows:

$$e_1 = \mathrm{X}_1 + \alpha_1^6 \mathrm{Y}_6 + \alpha_1^7 \mathrm{Y}_7 + \alpha_1^8 \mathrm{Y}_8 \quad \text{where } \alpha_1^7 \neq 0. \tag{3.24}$$

The polar space of E_1 is: $H(E_1) = \{\xi \in T_z\Sigma | (\varpi^a)_{a=1,\dots,5}(\xi) = (\varpi^6 \wedge \eta^1 + \varpi^7 \wedge \eta^2)(\xi, e_1) = (\varpi^7 \wedge \eta^1 - \varpi^8 \wedge \eta^2)(\xi, e_1) = (\varpi^6 \wedge \varpi^7 + \varpi^7 \wedge \varpi^8 - K\eta^1 \wedge \eta^2)(\xi, e_1) = 0\}$. Every tangent vector ξ satisfying the following system of equations belongs to the polar space of E_1: $\xi^1 = \dots = \xi^5 = \xi^6 - \alpha_1^6\xi_M^1 - \alpha_1^7\xi_M^2 = \xi^7 - \alpha_1^7\xi_M^1 - \alpha_1^8\xi_M^2 = \alpha_1^7\xi^6 - \alpha_1^6\xi^7 + \alpha_1^8\xi^7 - \alpha_1^7\xi^8 + \mathcal{K}\xi_M^2 = 0$. Therefore, $C_1 = 8$ and $r_1 = \dim H(E_1) - 2 = 2 - 2 = 0$. There then exists a 2-integral element of $\mathcal{I}$. Consider e_2 whose coordinates are solutions to the polar system and set $\xi_{\mathcal{M}}^1 = 0$ and $\xi_{\mathcal{M}}^2 = 1$. Hence,

$$e_2 = X_2 + \alpha_1^7 Y_6 + \alpha_1^8 Y_7 + \Big(\frac{(\alpha_1^7)^2 + (\alpha_1^8)^2 - \alpha_1^6\alpha_1^8 + \mathcal{K}}{\alpha_1^7}\Big) Y_8 \tag{3.25}$$

CODIMENSION OF $\mathcal{V}_2(\mathcal{I}, \eta^1 \wedge \eta^2)$

As in the previous proof, one can easily check that $\mathrm{codim}\mathcal{V}_2(\mathcal{I}, \eta^1 \wedge \eta^2)$ in $G_2(T\Sigma, \eta^1 \wedge \eta^2)$ is 13. The 2-integral flag is ordinary since it passes the Cartan test, i.e., $C_0 + C_1 = 5 + 8 = 13 = \mathrm{codim}\mathcal{V}_2(\mathcal{I}, \eta^1 \wedge \eta^2)$. The Cartan–Kähler theorem assures the existence and the uniqueness of the integral manifold of $\mathcal{I}$. By construction, the isometric Lagrangian embedding depends on three arbitrary functions of one variable (α_1^6, α_1^7 and α_1^8). □

3.A LAGRANGIAN MANIFOLDS IN $\mathbb{R}^{2m}$

As for the dimension 4, one can easily expect that on a symplectic space $(\mathbb{R}^{2m}, dx^1 \wedge dx^m + \dots, dx^m \wedge dx^{2m})$, the vector space defined by $x^{m+1} = \dots x^{2m} = 0$ and which tangent space is $\mathrm{span}\{\partial/\partial x^1, \dots, \partial/\partial x^m\}$, is a Lagrangian space. As previously, we address the question of finding Lagrangian manifolds that are not necessarily flat.

Proposition 3.11 – Lagrangian manifolds in $\mathbb{R}^{2m}$ There indeed exist m-dimensional Lagrangian manifolds of the 2m-dimensional symplectic space.

Proof. Let $(\mathbb{R}^{2m}, \Omega)$ be a symplectic space, where $\Omega = dx^1 \wedge dx^{m+1} + \dots + dx^m \wedge dx^{2m}$. As for the dimension 4, $\{\omega\}$ is closed and since it does not contain neither functions nor differential 1-forms, every point and tangent vector of $T_M\mathbb{R}^{2m}$ are integral points and 1-integral elements of $\{\omega\}$ respectively. Hence, $C_0 = 0$. A tangent vector of $\mathbb{R}^{2m}$ is expressed as follows:

$$\xi = \xi^1\frac{\partial}{\partial x^1} + \dots + \xi^m\frac{\partial}{\partial x^m} + \xi^{m+1}\frac{\partial}{\partial x^{m+1}} + \dots \xi^{2m}\frac{\partial}{\partial x^{2m}}. \tag{3.26}$$

Consider then, at a given point $M \in \mathbb{R}^{2m}$, the integral element $E_1 = \mathrm{span}\{e_1\}$, where $e_1 = \partial/\partial x^1 + \alpha_1^{m+1}\partial/\partial x^{m+1} + \dots + \alpha_1^{2m}\partial/\partial x^{2m}$. The polar space of E_1 is $H(E_1) = \{\xi \in T_M\mathbb{R}^{2m} | \omega(\xi, e_1) = 0\}$, and tangent vectors satisfying $\alpha_1^{m+1}\xi^1 + \dots + \alpha_1^{2m}\xi^m - \xi^{m+1} = 0$ belong to $H(E_1)$. Thus $C_1 = 1$ and the extension rank is $r_1 = 2m-2$. There exist then 2-integral elements of $\mathbb{R}^{2m}$. Consider then $E_2 = \mathrm{span}\{e_1, e_2\}$ where $e_2 = \partial/\partial x^2 + \alpha_2^{m+1}\partial/\partial x^{m+1} + \dots + \alpha_2^{2m}\partial/\partial x^{2m}$ and $\alpha_2^{m+1} = \alpha_1^{m+2}$. The following considerations are the same for $\lambda = 2, \dots, m$. Consider $E_\lambda = \mathrm{span}\{e_1, \dots, e_\lambda\}$, where

$$e_\lambda = \frac{\partial}{\partial x^\lambda} + \alpha_\lambda^{m+1}\frac{\partial}{\partial x^{m+1}} + \dots + \alpha_\lambda^{2m}\frac{\partial}{\partial x^{2m}} \quad \text{and} \quad \alpha_\lambda^{m+\nu} = \alpha_\nu^{m+\lambda} \text{ for } \nu = 1, \dots, \lambda - 1. \tag{3.27}$$

Therefore, the polar space of E_λ is $H(E_\lambda) = \{\xi \in T_M\mathbb{R}^{2m} | \omega(\xi, e_1) = \dots = \omega(\xi, e_\lambda) = 0\}$, and tangent vectors that satisfy the system $\alpha_\nu^{m+1}\xi^1 + \dots + \alpha_\nu^{2m}x^m - \xi^{m+\nu}$, for $\nu = 1, \dots \lambda$, belong to $\mathrm{H}(\mathrm{E}_\lambda)$. Thus, $\mathrm{C}_\lambda = \lambda$, and the extension rank is $r_\lambda = 2m - \lambda$. There exist then λ-integral elements of $\{\omega\}$. Except for $\lambda = m-1$, consider then $\mathrm{E}_{\lambda+1} = \mathrm{span}\{e_1, \ldots, e_{\lambda+1}\}$ where $e_{\lambda+1}$ is defined as in 3.27.

CODIMENSION OF $\mathcal{V}_m(\mathcal{I}, \eta^\Lambda)$

Consider a basis of the Grassmannian $\mathrm{G}_m(\mathrm{T_M}\mathbb{R}^{2m})$ defined by $\mathrm{X}_i(\mathrm{E}) = \partial/\partial x^i + \mathrm{P}_i^{m+1}\partial/\partial x^{m+1} + \cdots + \mathrm{P}_i^{2m}\partial/\partial x^{2m}$, for $i = 1, \ldots, m$. The vanishing of the pull-back of the symplectic form on the Grassmannian $\mathrm{G}_m(\mathrm{T_M}\mathbb{R}^{2m})$ leads to the following system: $\mathrm{P}_j^{i+m} - \mathrm{P}_i^{j+m} = 0$ where $i, j = 1, \ldots, m$. The differentials of these function are linearly independent and hence, the codimension of $\mathcal{V}_m(\mathcal{I}, \mathrm{d}x^1 \wedge \wedge \mathrm{d}x^m)$ is $m(m+1)/2$. The sum of the characters is also $m(m+1)/2$. Therefore, the Lagrangian EDS passes the Cartan test and consequently, ω is in involution. Finally, the Cartan–Kähler theorem assures the existence of an integral manifold of ω. $\square$

3.B THE CARTAN–JANET THEOREM

We now state and prove the Cartan–Janet theorem concerning local isometric embedding of a real analytic Riemannian manifold. Schlaefli in his paper in 1871 [Sch71] conjectured that an m-dimensional Riemannian manifold can always be locally embedded in an $\mathrm{N} = \frac{1}{2}m(m+1)$ dimensional Euclidean space. In 1926, Janet [Jan26] proved the result for the dimension 2 by resolving a differential system and explaining how we get the result in the general case. In 1927, Élie Cartan [Car27] gave the complete proof of the result. His method is based on his theory of involutive Pfaffian systems. Later, in 1931, Burstin [Bur31] generalized Janet's method and obtained the result in the general case.

Theorem 3.12 – Cartan–Janet Every m-dimensional real analytic Riemannian manifold can be locally embedded isometrically in an $m(m+1)/2$-dimensional Euclidean space.

Proof. The details of the computations can be found in [Kah06, Kah08a]. The proof can be divided into six steps:

STEP 1. Let $(\mathcal{M}^m, g)$ be an m-dimensional real analytic Riemannian manifold, where g is a Riemannian metric, i.e. a covariant symmetric positive defined 2-tensor, such that at a given point M of $\mathcal{M}^m$, g_{M} reduces in a orthonormal basis to the identity matrix. However in a open neighborhood of M, the matrix of g can not always be the identity yet it can always be reduced to the diagonal matrix $g = g_{11}\mathrm{d}x^1 \otimes \mathrm{d}x^1 + g_{22}\mathrm{d}x^2 \otimes \mathrm{d}x^2 + \cdots + g_{mm}\mathrm{d}x^m \otimes \mathrm{d}x^m$, where the terms g_{ii} are positive functions such that $g_{ii} = 1$ at M. We denote $\eta^i = \sqrt{g_{ii}}\mathrm{d}x^i$ and thus g can be written as follows:

$$g = \eta^1 \otimes \eta^1 + \eta^2 \otimes \eta^2 + \cdots + \eta^m \otimes \eta^m. \tag{3.28}$$

$\eta = (\eta^1, \eta^2, \ldots, \eta^m)$ is then an orthonormal coframe in the neighborhood of $\mathrm{M} \in \mathcal{M}$ which satisfies Cartan's structure equations $\mathrm{d}\eta^i + \eta^i_j \wedge \eta^j = 0$ and $\mathrm{d}\eta^i_j + \eta^i_k \wedge \eta^k_j = \Omega^i_j$ where (η^i_j) is the matrix of 1-form of the Levi-Civita connection on $\mathcal{M}$ and (Ω^i_j) is the curvature 2-form of the connection. Note that indices i, j and k vary from 1 to $m = \dim\mathcal{M}^m$.

STEP 2. Let $\mathbb{E}^{\mathrm{N}}$ be an N-dimensional Euclidean space (for the moment, $\mathrm{N} > m$) endowed with the usual scalar product $\langle , \rangle_{\mathbb{E}^{\mathrm{N}}}$. Let us consider $\mathcal{F}(\mathbb{E}^{\mathrm{N}})$ to be a positively-oriented orthonormal frame bundle on $\mathbb{E}^{\mathrm{N}}$. In what follows, we will not work on the entire bundle $\mathcal{F}(\mathbb{E}^{\mathrm{N}})$, but rather on a quotient $\mathcal{F}_m(\mathbb{E}^{\mathrm{N}})$. An element in $\mathcal{F}_m(\mathbb{E}^{\mathrm{N}})$ has the form $(x; e_1, e_2, \ldots, e_m)$, where $x \in \mathbb{E}^{\mathrm{N}}$ and $(e_1, e_2, \ldots, e_m)$ is a positively-oriented orthonormal set of vectors in $\mathbb{E}^{\mathrm{N}}$. Since any such a frame $(e_1, \ldots, e_m)$ can be completed in an oriented orthonormal frame of $\mathbb{R}^{\mathrm{N}}$, $\mathcal{F}_m(\mathbb{E}^{\mathrm{N}})$ is diffeomorphic to $\mathbb{E}^{\mathrm{N}} \times \mathrm{SO(N)}/\mathrm{SO}(\mathrm{N}-m)$ and hence of dimension $\mathrm{N}(m+1) - m(m+1)/2$. On $\mathcal{F}_m(\mathbb{E}^{\mathrm{N}})$, we define a set of 1-forms ω^α and ω^α_β by: $\mathrm{d}x = \omega^{\mathrm{A}} e_{\mathrm{A}}$ and $\mathrm{d}e_{\mathrm{B}} = \omega^{\mathrm{A}}_{\mathrm{B}} e_{\mathrm{A}}$, where the indices A, B and C vary from 1 to N. Therefore $(\omega^1, \ldots, \omega^m, \omega^{m+1}, \ldots, \omega^{\mathrm{N}})$ form an orthonormal coframe of $\mathcal{F}(\mathbb{E}^{\mathrm{N}})$. Furthermore, Cartan's structure equations on $\mathcal{F}_m(\mathbb{E}^{\mathrm{N}})$ are $\mathrm{d}\omega^{\mathrm{A}} + \omega^{\mathrm{A}}_{\mathrm{B}} \wedge \omega^{\mathrm{B}} = 0$ and $\mathrm{d}\omega^{\mathrm{A}}_{\mathrm{B}} + \omega^{\mathrm{A}}_{\mathrm{C}} \wedge \omega^{\mathrm{C}}_{\mathrm{B}} = 0$. Notice that $(\omega^{\mathrm{A}}_{\mathrm{B}})$ is the $\mathrm{N} \times \mathrm{N}$ skew-symmetric matrix connection form of the Levi-Civita connection on $\mathbb{E}^{\mathrm{N}}$.

Step 3. Let us consider the product manifold $\mathcal{M} \times \mathcal{F}_m(\mathbb{E}^{\mathrm{N}})$. Let $\mathcal{I}_0$ be the exterior ideal on $\mathcal{M} \times \mathcal{F}_m(\mathbb{E}^{\mathrm{N}})$ generated by the Pfaffian system $\mathrm{I}_0 = \{(\omega^i - \eta^j), \omega^a\}$, where the indices a, b and c vary from $m+1$ to N.

Proposition 3.13 Every m-dimensional integral manifold of $\mathcal{I}_0$ on which the form $\Delta = \omega^1 \wedge \cdots \wedge \omega^m$ does not vanish is locally the graph of a function $f : \mathcal{M} \longrightarrow \mathcal{F}_m(\mathbb{E}^{\mathrm{N}})$ having the property that $u = \pi_{\mathbb{E}^{\mathrm{N}}} \circ f : \mathcal{M} \longrightarrow \mathbb{E}^{\mathrm{N}}$ is a local isometric embedding, where $\pi_{\mathbb{E}^{\mathrm{N}}}$ is the projection of $\mathcal{F}_m(\mathbb{E}^{\mathrm{N}})$ on the Euclidean space $\mathbb{E}^{\mathrm{N}}$. Conversely, every local isometric embedding $u : \mathcal{M} \longrightarrow \mathbb{E}^{\mathrm{N}}$ arises in a unique way from this construction.

Step 4. According to proposition 3.13, the existence of an integral manifold of $\mathcal{I}_0$ for which Δ is non zero, is a neccessary condition for the existence of a local isometric embedding. However, the theorems and the results that we discussed deal with closed exterior differential systems. Therefore it is natural to add to the Pffafian system I_0 the exterior differentiation of each 1-form and hence, we obtain a closed exterior differential system: $\mathrm{I}_0 \cup \mathrm{dI}_0$. When we compute the exterior differentiation of $(\omega^i - \eta^i)$, we remark new differential forms and an interesting result :

$$\mathrm{d}(\omega^i - \eta^i) = -(\omega^i_j - \eta^i_j) \wedge \omega^j = 0. \tag{3.29}$$

By Cartan's lemma, $\omega^i_j - \eta^i_j = h^i_{jk}\omega^k$, with $h^i_{jk} = h^i_{kj} = -h^j_{ik}$. With the symmetry and the skew-symmetry of the functions h^i_{jk}, we conclude that h^i_{jk} are zero and so, $\omega^i_j - \eta^i_j = 0$.

Remark 3.14 – Geometric interpretation. The vanishing of the forms $\omega^i_j - \eta^i_j = 0$ implies that $f^*(\omega^i_j) = \eta^i_j$ where f is the function of proposition 3.13, which means that the pull-back of the Levi-Civita connection by an isometric embedding is the Levi-Civita connection on $\mathcal{M}$.

Therefore, we extend the exterior differential I_0 and obtain an exterior differential system on $\mathcal{M} \times \mathcal{F}_m(\mathbb{E}^{\mathrm{N}})$ $\mathrm{I}_1 = \{(\omega^i - \eta^i)_{i=1,\ldots,m}, (\omega^a)_{a=m+1,\ldots,\mathrm{N}}, (\omega^i_j - \eta^i_j)_{1 \le i < j \le m}\}$. In order to have a closed one, we add the exterior differentiation of each form, and we denote $\mathcal{I}$ the exterior differential ideal generated by I_1. Instead of looking for an integral manifold of $\mathcal{I}_0$, we will look for the existence of an integral manifold of $\mathcal{I}$. From the structure equations stated earlier, we obtain, modulo I_1, the following results:

$$\mathrm{d}(\omega^i - \eta^i) \equiv 0 \quad , \quad \mathrm{d}\omega^a \equiv -\omega^a_i \wedge \omega^i \quad \text{and} \quad \mathrm{d}(\omega^i_j - \eta^i_j) \equiv -(\omega^i_a \wedge \omega^a_j + \Omega^i_j). \tag{3.30}$$

On $\mathcal{X}$, the integral manifold of $\mathcal{I}$, $\omega^a = 0$, and thus $\mathrm{d}\omega^a = 0$. We conclude that $\omega^a_i \wedge \omega^i = 0$. The Cartan lemma 1.17 assures the existence of m^2 functions h^a_{ij} such that $\omega^a_i = h^a_{ij}\omega^j$, where $h^a_{ij} = h^a_{ji}$. We can then write: $\omega^a_i - h^a_{ij}\omega^j = 0$ on $\mathcal{X}$. However, nothing lead us to believe that this equality will be true outside of $\mathcal{X}$. We then define the differential 1-form π_i^a on $\mathcal{M}\times\mathcal{F}_m(\mathbb{E}^{\mathrm{N}})$ as follows

$$\pi_i^a = \omega_i^a - h_{ij}^a\omega^j \tag{3.31}$$

On $\mathcal{X}$, $\omega_j^i - \eta_j^i = 0$, and thus $\mathrm{d}(\omega_j^i - \eta_j^i) = 0$. Restricted to $\mathcal{X}$, the last equation of (3.30) becomes $\omega_a^i \wedge \omega_j^a + \Omega_j^i = 0$. Using (3.31), we can establlish the Gauss equation as follows:

$$\sum_{a=m+1}^{\mathrm{N}} (h_{ik}^a h_{jl}^a - h_{il}^a h_{jk}^a) = \mathcal{R}_{ijkl}. \tag{3.32}$$

We see that the exterior differential system $\tilde{\mathrm{I}} = \{(\omega^i - \eta^i), \omega^a, (\omega_j^i - \eta_j^i), \pi_i^a\}$ when the Gauss equation is satisfied, generates the exterior differential ideal $\mathcal{I}$. Looking for integral elements of I is equivalent to looking for integral elements of $\tilde{\mathrm{I}}$ for which the Gauss equation is satisfied. We shall proceed with this in the following steps. Moreover, $\tilde{\mathrm{I}}$ contains less differential 1-forms than the exterior differential system I.

STEP 5. The functions h_{ij}^a are symmetric in their two low indices. If we consider an $(\mathrm{N}-m)$-dimensional euclidean space $\mathcal{W}$, then the matrix (h_{ij}^a) can be viewed as a symmetric element of $\mathbb{R}^m(i,j = 1,\ldots,m)$ taking value in $\mathcal{W}$, i.e. $(h_{aij}) \in \mathcal{W}\otimes \mathrm{S}^2(\mathbb{R}^m)$.

Proposition 3.15 Let $\mathcal{K}_m$ be the set of Riemannian curvature tensors $\mathcal{R}$ defined as follows: $\mathcal{K}_m = \{(\mathbb{R}_{ijkl}) \in \mathrm{S}^2(\wedge^2\mathbb{R}^m) | \mathcal{R}_{ijkl} + \mathcal{R}_{kijl} + \mathcal{R}_{jkil} = 0\}$. Then $\dim\mathcal{K}_m = m^2(m^2-1)/12$.

With these considerations, we have the following lemma:

Lemma 3.16 Suppose that $\kappa = \mathrm{N} - m \geq m(m-1)/2$. Let $\mathcal{H} \subset \mathcal{W}\otimes \mathrm{S}^2(\mathbb{R}^m)$ be an open set containing the elements $h = (h_{ij})$ such that the vectors $\{h_{ij} | 1 \leq i \leq j \leq m-1\}$ are linearly independent as elements of $\mathcal{W}$. The map $\gamma : \mathcal{H} \longrightarrow \mathcal{K}_m$, that associates $h \in \mathcal{H}$ with $\gamma(h) \in \mathcal{K}_m$ such that $\Big(\gamma(h)\Big)_{ijkl} = h_{ik}.h_{jl} - h_{il}h_{jk}$, is a surjective submersion.

STEP 6. Finally, we want to show the existence of an m-dimensional ordinary integral element. Let us recall that the exterior ideal $\mathcal{I}$ of $\mathcal{M}\times\mathcal{F}_m(\mathbb{E}^{\mathrm{N}})$ is generated by $s = \mathrm{N}(m+1) - m(m+1)/2$ 1-forms $\{(\omega^i - \eta^i), (\omega^a), (\omega_j^i - \eta_j^i), (\pi_i^a)\}$. Let $\mathcal{Z} = \{(\mathrm{M}, \Upsilon, h) \in \mathcal{M}\times\mathcal{F}_m(\mathbb{E}^{\mathrm{N}}) \times \mathcal{H} | \gamma(h) = \mathcal{R}(\mathrm{M})\}$. $\mathcal{Z}$ is a submanifold (the fiber of $\mathcal{R}$ by a submersion and the surjectivity of γ ensures that $\mathcal{Z} \neq \emptyset$). We define the map $\Phi : \mathcal{Z} \longrightarrow \mathcal{V}_m(\mathcal{I}, \Delta)$ that associates $(\mathrm{M}, \Upsilon, h)$ with the m-plane at (M, Υ) annihilated by the 1-forms that generate $\mathcal{I}$(the exterior differential system $\tilde{\mathrm{I}}$). The map Φ is an embedding and so $\Phi(\mathcal{Z})$ is a submanifold of $\mathcal{V}_m(\mathcal{I}, \Delta)$. We will show that $\Phi(\mathcal{Z})$ contains only ordinary integral elements. Let $(\mathrm{M}, \Upsilon, h) \in \mathcal{Z}$ be a point. Denote by $\mathrm{E} = \Phi(\mathrm{M}, \Upsilon, h)$ the integral element defined as follows: $\mathrm{E} = \{v \in \mathrm{T}_{(\mathrm{M},\Upsilon)}(\mathcal{M}\times\mathcal{F}_m(\mathbb{E}^{\mathrm{N}})) | (\omega^i - \eta^i)(v) = \omega^a(v) = (\omega_j^i - \eta_j^i)(v) = \pi_i^a(v) = 0\}$. Therefore, E is an m-dimensional integral element. As a matter of fact, s is the number of differential forms that generate the ideal $\mathcal{I}$ and $\dim\mathcal{M}\times\mathcal{F}_m(\mathbb{E}^{\mathrm{N}}) - m = s$. We will apply the proposition 2.34. Let $\mathcal{I}$ be the exterior ideal of $\mathcal{M}\times\mathcal{F}_m(\mathbb{E}^{\mathrm{N}})$ as defined above. This ideal does not contain any 0-forms . $\mathrm{E} \in \mathcal{V}_m(\mathcal{I})$ at $(\mathrm{M}, \Upsilon) \in \mathcal{M}\times\mathcal{F}_m(\mathbb{E}^{\mathrm{N}})$. Let $\omega^i, (\omega^i - \eta^i), \omega^a, (\omega_j^i - \eta_j^i), \pi_i^a$ be a coframe of $\mathcal{M}\times\mathcal{F}_m(\mathbb{E}^{\mathrm{N}})$ in the neighborhood of (M, Υ) such that $\mathrm{E} = \{v \in \mathrm{T}_{\mathrm{M},\Upsilon}(\mathcal{M}\times\mathcal{F}_m(\mathbb{E}^{\mathrm{N}})) | (\omega^i - \eta^i)(v) = \omega^a(v) = (\omega_j^i - \eta_j^i)(v) = \pi_i^a(v) = 0\}$. Finally, the characters c_p, which are the codimension of $\mathrm{H}(\mathrm{E}_p)$ in $\mathrm{G}_m(\mathrm{T}(\mathcal{M}\times\mathcal{F}_m\mathbb{E}^{\mathrm{N}}))$, are equal to $\mathrm{C}_p = \mathrm{N} + m(m-1)/2 + (\mathrm{N}-m)p + mp(m-p)/2$. Since Φ is an embedding, $\dim\Phi(\mathcal{Z}) = \dim\mathcal{Z}$, and hence

$$\dim\mathrm{G}_m(\mathrm{T}(\mathrm{M}\times\mathfrak{U})) - \dim\Phi(\mathcal{Z}) = \mathrm{N}m(m+1)/2 + m^2(m^2-1)/12. \tag{3.33}$$

We conclude that the codimension of $\Phi(\mathcal{Z})$ in $\mathrm{G}_m\Big(\mathrm{T}(\mathrm{M}\times\mathfrak{U})\Big)$ is equal to $\mathrm{C}_0+\mathrm{C}_1+\cdots+\mathrm{C}_{m-1}$. By Cartan's test, $\mathrm{E}\in\mathcal{V}_m(\mathcal{I},\Omega)$ is an ordinary integral element of $\mathcal{I}$. The Cartan–Kähler theorem (corollary 2.37.) ensures the existence of an integral manifold $\mathcal{X}$ passing through (M,Υ) and having E as a tangent space at (M,Υ). In particular, $\mathrm{E}\in\mathcal{V}_m(\mathcal{I}_0,\Omega)$. By proposition 3.13, there then exists an isometric embedding of $(\mathcal{M}^m,g)$ in $(\mathbb{E}^{\mathrm{N}},\langle,\rangle_{\mathbb{E}^{\mathrm{N}}})$. $\square$

Chapter 4

On generalized isometric embeddings

The purpose of this chapter is to introduce the generalized isometric embedding problem and to use it for the construction of conservation laws for a certain class of PDEs. First, the "usual" definition of conservation laws with tangent vector fields is given. Then, using a Riemannian metric, another definition which uses differential forms is expounded. After stating the generalized isometric embedding problem, we present two main motivations: the isometric embedding problem of Riemannian manifolds and harmonic maps between Riemannian manifolds. Next, all of the established generalized isometric embedding results are stated. Finally, we present an application to covariant divergence-free energy-momentum tensors.

4.1 Conervation laws

Many fundamental quantities in physics, for instance: mass, energy, movement quantity, momentum, electric charge, etc. when some conditions are satisfied, do not change as the physical system evolves. Since these quantities are preserved, one can then consider that there are *conservation laws* that govern the evolution of a given physical system. A mathematical definition of conservation laws is as follows:

Definition 4.1 – Conservation laws via vector fields Let $(\mathcal{M}, g)$ be an m-dimensional Riemannian manifold and let $\mathcal{F}$ be either a function space or a cross-section space. A conservation law is a mapping from $\mathcal{F}$ to the tangent vector fields such that the solutions to a given PDE are mapped to divergence-free tangent vector fields.

Using the Riemannian metric g, one can canonically associate each vector field $\mathrm{X} \in \Gamma(\mathrm{T}\mathcal{M})$ with a differential 1-form $\alpha_{\mathrm{X}} := g(\mathrm{X}, \cdot)$. The divergence of a tangent vector field is a function, and can be defined in two (equivalent) ways:

$$\begin{aligned} \mathrm{div}(\mathrm{X}) &= *\mathrm{d} * \alpha_{\mathrm{X}} && (4.1)\\ \mathrm{div}(\mathrm{X})\mathrm{vol}_{\mathcal{M}} &= \mathrm{d}(\mathrm{X} \lrcorner\, \mathrm{vol}_{\mathcal{M}}) && (4.2) \end{aligned}$$

where $*$ is the Hodge operator, $\mathrm{vol}_{\mathcal{M}}$ is the volume form on $\mathcal{M}$, and $\mathrm{X} \lrcorner\, \mathrm{vol}_{\mathcal{M}}$ is the interior product of $\mathrm{vol}_{\mathcal{M}}$ by the vector field X. In (4.2), the requirement $\mathrm{div}(\mathrm{X}) = 0$ may be replaced by the requirement $\mathrm{d}(\mathrm{X} \lrcorner\, \mathrm{vol}_{\mathcal{M}}) = 0$, leading to another possible definition of conservation laws:

Definition 4.2 – Conservation laws via differential forms Let $(\mathcal{M}, g)$ be an m-dimensional Riemannian manifold and let $\mathcal{F}$ be either a function space or a cross section space. Then a conservation law is a mapping from $\mathcal{F}$ to differential $(m-1)$-forms such that the solutions to a given PDE are mapped to closed differential $(m-1)$-forms.

More generally, we could extend the notion of conservation laws as mapping to differential p-forms. For instance, Maxwell equations in vacuum can be expressed, as it is well-known, by requiring a system of differential 2-forms to be closed.

4.2 The generalized isometric embedding problem

Originally formulated by Hélein [Hél96], the following problem addresses the question of finding conservation laws for a class of PDEs described as follows:

Problem 4.3 – The generalized isometric embedding problem Let $\mathbb{V}$ be an n-dimensional vector bundle over $\mathcal{M}$. Let g be a metric bundle and ∇ a connection that is compatible with that metric. We then have a covariant derivative d_∇ acting on vector bundle valued differential forms. Assume that ϕ is a given covariantly closed $\mathbb{V}$-valued differential p-form on $\mathcal{M}$, i.e.,

$$\mathrm{d}_\nabla \phi = 0. \tag{4.3}$$

Does there exist $\mathrm{N} \in \mathbb{N}$ and an embedding Ψ of $\mathbb{V}$ into $\mathcal{M} \times \mathbb{R}^{\mathrm{N}}$ given by $\Psi(x, \mathrm{X}) = (x, \Psi_x \mathrm{X})$, where Ψ_x is a linear map from $\mathbb{V}_x$ to $\mathbb{R}^{\mathrm{N}}$ such that:

- Ψ is isometric, i.e, for every $x \in \mathcal{M}$, the map $\Psi_x : \mathbb{V}_x \longrightarrow \mathbb{R}^{\mathrm{N}}$ is an isometry,
- If $\Psi(\phi)$ is the image of ϕ by Ψ, i.e., $\Psi(\phi)_x = \Psi_x \circ \phi_x$ for all $x \in \mathcal{M}$, then

$$\mathrm{d}\Psi(\phi) = 0. \tag{4.4}$$

In this problem, the equation (4.3) represents the given PDE (or a system of PDEs) and equation (4.4) plays the role of a conservation law. Note that there is no particular structure on the base manifold.

Remark 4.4 – The line bundle case. The generalized isometric embedding problem is trivial when the vector bundle is a line bundle (n=1). Indeed, the only connection on a real line bundle that is compatible with the metric is the flat one.

4.3 Motivations

As expounded in [Hél96], there are basically two main motivations to the statement of the above problem. The first motivation is the isometric embedding problem of Riemannian manifolds. We start by recalling the definition of Riemannian isometry between Riemannian manifolds and state the isometric embedding problem. Then we show how the generalized isometric embedding problem is related to the isometric embedding and state an important local isometric embedding result, which solves the generalized problem in this specific case. The second main motivation is related to harmonic maps between Riemannian manifolds. After defining harmonics maps and presenting some examples, we explain how harmonics maps can be characterized by using the generalized isometric embedding problem's ingredients.

4.3.1 The isometric embedding problem

A fundamental example is the isometric embedding of Riemannian manifolds in Euclidean spaces.

Definition 4.5 Let $(\mathcal{M}^m, g)$ and $(\mathcal{N}^n, h)$ be two Riemannian manifolds of dimensions m and n respectively. A map u defined on $(\mathcal{M}^m, g)$ with values in $(\mathcal{N}, h)$ is a Riemannian isometry if $u^*(h) = g$.

After the emergence of the abstract notion of manifolds, due to the works of Gauss [Gau27] and Riemann[Rie68], a natural question arose: does there exist an abstract manifold? Another way to express this question: is it possible that any given abstract manifold is in fact a submanifold of a certain Euclidean space? Or equivalently, does any arbitrary Riemannian manifold admit an isometric embedding in a Euclidean space? This problem is known as the isometric embedding problem, and has been considered in various specializations and with assorted conditions. It is related to the generalized isometric embedding problem as follows:

Isometric embedding vs generalized isometric embedding

The generalized isometric embedding problems' ingredients are: the vector bundle $\mathbb{V}$ over a manifold $\mathcal{M}$, a metric bundle g, a metric connection ∇ and a covariantly closed differential p-form ϕ. Denote these ingredients by the 5-tuple $(\mathbb{V}^n, \mathcal{M}^m, g, \nabla, \phi)_p$, which plays a central role in the puzzles and the upstairs geometries [Hél09]. A Riemannian manifold $(\mathcal{M}^m, g)$ then provides the base manifold, the vector bundle, and the metric, i.e., $\mathbb{V} = \mathrm{T}\mathcal{M}$. There is also a natural connection ∇ on $\mathrm{T}\mathcal{M}$ which is the Levi-Civita connection. The remaining ingredient, i.e., a covariantly closed differential form is given as follows: Let ϕ be the identity map on $\mathrm{T}\mathcal{M}$. Then $\phi = \mathrm{Id}_{\mathrm{T}\mathcal{M}}$ can be seen as a $\mathrm{T}\mathcal{M}$-valued differential 1-form, and it turns out, as explained later in the theorem 4.12's proof, that:

Proposition 4.6 – Canonical covariantly closed 1-form on a Riemannian manifold Let $(\mathcal{M}^m, g)$ be an m-dimensional Riemannian manifold and ∇ a g-compatible connection on $\mathrm{T}\mathcal{M}$. Then $\mathrm{Id}_{\mathrm{T}\mathcal{M}}$ is a covariantly closed differential 1-form, i.e., $\mathrm{d}_\nabla(\mathrm{Id}_{\mathrm{T}\mathcal{M}}) = 0$, if and only if the connection ∇ is torsion-free.

Since the connection ∇ is already g-compatible, it is equivalent to say that the identity map on the tangent bundle $\mathrm{T}\mathcal{M}$ is a covariantly closed $\mathrm{T}\mathcal{M}$-valued differential 1-form if and only if ∇ is the Levi-Civita connection. Actually, any solution to the generalized isometric embedding in the $(\mathrm{T}\mathcal{M}, \mathcal{M}, g, \nabla, \mathrm{Id}_{\mathrm{T}\mathcal{M}})$ case provides an isometric embedding u of the Riemannian manifold $\mathcal{M}$ into a Euclidean space $\mathbb{R}^{\mathrm{N}}$ through the integration of

$$\mathrm{d}u = \Psi(\phi) \tag{4.5}$$

and conversely. An answer to the local analytic isometric embedding of Riemannian manifolds is given by the Cartan–Janet theorem (theorem 3.12). Nash[Nas56] solved the isometric embedding problem in the smooth and global case. Despite the fact that the Cartan–Janet result is local and the analycity hypotheses on the data may seem to be too restrictive, the Cartan–Janet theorem is important because it actualizes the embedding in an optimal dimension.

Consequently, if the generalized isometric embedding problem has a positive answer for $p = 1$, the notion of isometric embeddings of Riemannian manifolds is extended to the notion of *generalized isometric embeddings* of vector bundles. The general problem, when p is arbitrary, can also be viewed as an *embedding of covariantly closed vector bundle valued differential p-forms.*

4.3.2 HARMONIC MAPS BETWEEN RIEMANNIAN MANIFOLDS

Harmonics maps between Riemannian manifolds provide the ingredients of examples of generalized isometric problem. For that purpose, we give a brief introduction to harmonic maps and present some examples. The reader may also refer to [HW08] for a characterization of harmonic maps by *tension fields*, which is not presented in this section, and also for extra examples. Finally, we explain how harmonic maps between Riemannian manifolds are related to the generalized isometric embedding of vector bundle and with conservations laws.

Definition 4.7 – Harmonic map between Riemannian manifolds Let $(\mathcal{M}^m, g)$ and $(\mathcal{N}^n, h)$ be two Riemannian manifolds of dimension m and n respectively. A map u defined on $(\mathcal{M}^m, g)$ with values in $(\mathcal{N}, h)$ is a harmonic map if u is a critical point of the Dirichlet functional

$$\mathrm{E}[u] = \int_{\mathcal{M}} \frac{|\mathrm{d}u|^2}{2} \mathrm{vol}_{\mathcal{M}} \tag{4.6}$$

where $\mathrm{vol}_{\mathcal{M}}$ is the volume measure on $\mathcal{M}$ by the metric g, and $|\mathrm{d}u|^2$ is the Hilbert–Schmidt norm of $\mathrm{d}u$ given at a point $\mathrm{M} \in \mathcal{M}$.

Let us adopt the following convention: for $\mathcal{M}^m$, the indices α, β, γ vary form 1 to m, and for $\mathcal{N}^n$, the indices i, j, k vary from 1 to n. We also adopt the Einstein summation convention, i.e., there is a summation when the same index is repeated in high and low positions. In local coordinates $(x^1, \ldots, x^m)$ on $(\mathcal{M}, g)$, the volume measure $\mathrm{vol}_{\mathcal{M}}$ and the Hilbert–Schmidt norm of $\mathrm{d}u$ are expressed as follows:

$$\mathrm{vol}_{\mathcal{M}} = \sqrt{|g|}\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m \qquad \text{and} \qquad |\mathrm{d}u_{\mathrm{M}}|^2 = g^{ij}(\mathrm{M}) h_{\alpha\beta}(u(\mathrm{M})) \frac{\partial u^\alpha}{\partial x^i} \frac{\partial u^\beta}{\partial x^j}. \tag{4.7}$$

Critical points of the Dirichlet functional must satisfy the Euler–Lagrange system. Thus, an alternative local definition of harmonic maps is:

Proposition-Definition 4.8 – Harmonic maps A map u between two Riemannian manifolds $(\mathcal{M}^m, g)$ and $(\mathcal{N}^n, h)$ is harmonic if u satisfies:

$$\Delta_g u^i + g^{\alpha\beta} \Gamma^i_{jk}(u(\mathrm{M})) \frac{\partial u^j}{\partial x^\alpha} \frac{\partial u^k}{\partial x^\beta} = 0 \tag{4.8}$$

where Δ_g is the Laplacian operator on $(\mathcal{M}^m, g)$, $(g^{\alpha\beta})$ is the inverse metric of g, and the Γ^i_{jk} are the Christoffel symbols on the target manifold $(\mathcal{N}^n, h)$.

A way to understand physically harmonic maps is to imagine that the source Riemannian manifold $(\mathcal{M}, g)$ is made of rubber and the target Riemannian manifold $(\mathcal{N}, h)$ is made of marble. The shapes at rest of the manifolds are determined by their respective metrics. Then a map $u : \mathcal{M} \longrightarrow \mathcal{N}$ is a way to apply the rubber, which can be deformed, onto the marble, which can not. The energy $\mathrm{E}[u]$ then represents the total amount of elastic potential energy resulting from tension in the rubber. By definition, harmonic maps are the map u which minimizes the energy. Thus, one can imagine that harmonic maps are the ways of applying the rubber onto the marble such that, when one releases the rubber but still constrains it to stay everywhere in contact with the marble, the rubber is then actually itself in a position of equilibrium.

Harmonic maps are actually not that unusual in differential geometry, analysis and physics. From the above definition, one can easily guess some of the following examples:

Examples 4.9 – Harmonic maps. Let us consider a map $u : (\mathcal{M}, g) \longrightarrow (\mathcal{N}, h)$.

1. **Constant maps:** Let N be a fixed point in $\mathcal{N}$. A constant map $u : (\mathcal{M}, g) \longrightarrow (\mathcal{N}, h)$, that associates any point in $\mathcal{M}$ with the point $\mathrm{N} \in \mathcal{N}$, is naturally a harmonic map because the derivatives of u are identically zero, and so for the Laplacian of u. The equation (4.8) is then trivially satisfied.

2. **The identity map:** $\mathrm{Id}_{\mathcal{M}} : (\mathcal{M}, g) \longrightarrow (\mathcal{M}, g)$ is obviously a harmonic map. Using the above "physical explanation", if one applies a rubber manifold onto the same manifold made of marble to realize identity, then it is not hard to imagine that there is no tension at all on the rubber manifold.

3. **Harmonic functions:** If the target manifold $(\mathcal{N}^n, h)$ is the vector space $(\mathbb{R}, \langle , \rangle_{\mathbb{R}})$, then harmonic maps are nothing but harmonic functions on $(\mathcal{M}, g)$ because the equation (4.8) reduces to $\Delta_g u = 0$. Moreover, if the source manifold is an Euclidean space, namely $(\mathbb{R}^m, \langle , \rangle_{\mathbb{R}^m})$, then harmonic maps are (the usual) harmonic functions on $\mathbb{R}^m$, i.e., functions that satisfy $\Delta u = \partial^2 u/\partial x^1 \partial x^1 + \cdots + \partial^2 u/\partial x^m \partial x^m = 0$.

4. **Harmonic maps to Euclidean spaces:** When the target manifold $(\mathcal{N}^n, h)$ is an n-dimensional Euclidean space, namely $(\mathbb{R}^n, \langle , \rangle_{\mathbb{R}^n})$, then as one can expect, u is harmonic if each component u^i is a harmonic function on $(\mathcal{M}, g)$.

5. **Geodesics on a manifold:** Assume that the source manifold $(\mathcal{M}, g)$ is $(\mathbb{R}, \langle , \rangle_{\mathbb{R}})$. Let us consider t as a coordinate in $\mathbb{R}$. The equation (4.8) reduces then to:

$$\frac{\mathrm{d}^2 u^i}{\mathrm{d}t^2} + \Gamma^i_{jk}(u(\mathrm{M}))\frac{\mathrm{d}u^j}{\mathrm{d}t}\frac{\mathrm{d}u^k}{\mathrm{d}t} = 0 \tag{4.9}$$

   which represents the parameterization of a geodesic on the manifold $(\mathcal{N}, h)$. If the target manifold $(\mathcal{M}, g)$ is the circle $(\mathrm{S}^1, g_{\mathrm{S}^1})$, then parameterizations of closed geodesics are also harmonic maps.

6. **Holomorphic maps:** Consider a holomorphic or an antiholomorphic map $u : (\mathcal{M}, g, \mathrm{J}^{\mathcal{M}}) \longrightarrow (\mathcal{N}, g, \mathrm{J}^{\mathcal{N}})$ between two Kähler manifolds. The underlying real function is a harmonic [JE64]. To be convinced, consider $u : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that $u(x^1, x^2) = (u^1(x^1, x^2), u^2(x^1, x^2))$ satisfy the Cauchy–Riemann system (i.e., the complex underlying function is holomophic):

$$\begin{cases} \partial u^1/\partial x^1 - \partial u^2/\partial x^2 = 0 \\ \partial u^1/\partial x^2 + \partial u^2/\partial x^1 = 0. \end{cases} \tag{4.10}$$

   Since the source and target manifolds are flat, one can find local coordinate on which the Christoffel symbols vanish. Hence, u is a harmonic map if u^1 and u^2 are harmonic functions. This can be easily checked from the Cauchy–Riemann system given the fact that the second partial derivatives commute by the Schwarz lemma.

As shown by these examples, harmonic maps appear in many areas of geometry and analysis. But how are they related to the generalized isometric embedding problem? It turns out that harmonic maps produce the ingredients of the generalized isometric embedding problem: in other words, by using harmonic maps between Riemannian manifolds, one can produce, as expounded in [Hél96], a vector bundle over a manifold, a metric bundle, metric connexion, and, more importantly, a covariantly closed differential form.

Definition 4.10 – The pull-back bundle Let $\mathcal{M}$ and $\mathcal{N}$ be two manifolds of dimension m and n respectively, and let u be a map from $\mathcal{M}$ to $\mathcal{N}$. Then the pull-back bundle by u over $\mathcal{M}$ is $u^*(\mathrm{T}\mathcal{N}) = \{(x, \mathrm{X}) | x \in \mathcal{M} \text{ and } \mathrm{X} \in \mathrm{T}_{u(x)\mathcal{N}}\}$.

Proposition 4.11 – Characterization of harmonic maps by covariantly closed forms Let u be a map from an m-dimensional manifold $\mathcal{M}$ to an n-dimensional Riemannian manifold $(\mathcal{N}, h)$. Consider on $\mathcal{M}$ to be the pull-back bundle $u^*\mathrm{T}\mathcal{N}$ endowed with the pull-back metric $g = u^*(h)$ and the pull-back connection $\nabla = u^*\nabla^{\mathcal{N}}$, where $\nabla^{\mathcal{N}}$ is the Levi-Civita connection on $(\mathcal{N}, h)$. Then the $u^*\mathrm{T}\mathcal{N}$-valued differential $(m-1)$-form $*\mathrm{d}u$ is covariantly closed, i.e., $\mathrm{d}_\nabla(*\mathrm{d}u) = 0$, if and only if u is harmonic.

A positive answer to the generalized isometric embedding problem in this case will make it possible to construct conservation laws on $\mathcal{M}$ from covariantly closed vector bundle valued differential $(m-1)$-forms, provided, for example, by harmonic maps.

In his book [Hél96], motivated by the question of the compactness of weakly harmonic maps in Sobolev spaces in the weak topology (which remains an open question), Hélein considers harmonic maps between Riemannian manifolds, explains how conservation laws may be obtained explicitly by Noether's theorem if the target manifold is symmetric, and formulates the problem for non-symmetric target manifolds, which is in fact the generalized isometric embedding problem stated above.

4.4 On generalized isometric embedding results

We state in this section the different positive answers to the generalized isometric embedding problem. The first is related to the conservation law case, i.e., $p = m-1$ and is used in the next section for constructing conservation laws for covariant divergent-free energy-momentum tensors on a real analytic Riemannian manifold.

Theorem 4.12 – Local conservation laws by generalized isometric embeddings [Kah08b] Let $\mathbb{V}$ be a real analytic n-dimensional vector bundle over a real analytic m-dimensional manifold $\mathcal{M}$ endowed with a metric g and a connection ∇ compatible with g. Given a non-vanishing covariantly closed $\mathbb{V}$-valued differential $(m-1)$-form ϕ, there exists a local isometric embedding of $\mathbb{V}$ in $\mathcal{M} \times \mathbb{R}^{n+\kappa^n_{m,m-1}}$ over $\mathcal{M}$, where $\kappa^n_{m,m-1} \geqslant (m-1)(n-1)$ such that the image of ϕ is a conservation law.

For the remaining case, i.e., for $p = 1, \ldots, m-2$, the problem is still open in general. The following result is a positive answer to the covariantly closed vector bundle valued 1-form when the rank of the vector bundle is $n = 2$.

Theorem 4.13 – $(\mathbb{V}^2, \mathcal{M}^m, g, \nabla, \phi)_1$ case Let $\mathbb{V}^2$ be a real analytic 2-dimensional vector bundle over a real analytic m-dimensional manifold $\mathcal{M}$ endowed with a metric g and a connection ∇ compatible with g. Given a non-vanishing covariantly closed non-degenerate $\mathbb{V}$-valued differential 1-form ϕ, there exists a local isometric embedding of $\mathbb{V}^2$ in $\mathcal{M} \times \mathbb{R}^{n+\kappa^2_{m,1}}$ over $\mathcal{M}$, where $\kappa^n_{m,m-1} \geqslant 1$ such that the image of ϕ is a conservation law.

For an arbitrary n and m and for a non-degenerate ϕ, the same type of result is explained in chapter 6. This corresponds to the case of ϕ is bijective, injective, surjective, or more generally, of constant rank, as in [Hél09].

The following theorem is a positive answer to the generalized isometric embedding for $p = 2$ and for a vector bundle of rank 3 over a 4-dimensional manifold, which is a crucial dimension in physics. It is important not only because it gives an example for the case of a 2-form, but also because it provides an example of a 1-puzzle in the upstairs geometry [Hél09].

Theorem 4.14 – Generalized isometric embedding of 2-form with ASD condition Let $\mathcal{M}^4$ be an oriented real analytic 4-dimensional manifold endowed with a metric (actually a conformal structure is enough). Consider a real analytic vector bundle $\mathbb{V}^3$ of rank 3 over $\mathcal{M}^4$, endowed with a Riemannian metric g, an anti-self-dual g-compatible connection ∇, and a covariantly closed $\mathbb{V}^3$-valued differential 2-form ϕ of the form (6.3). There exists then a generalized isometric embedding Ψ of $\mathbb{V}^3$ into $\mathcal{M}^4 \times \mathbb{R}^{3+\kappa^3_{4,2,\mathrm{ASD}}}$, where $\kappa^3_{4,2,\mathrm{ASD}} \geqslant 4$, such that $\Psi(\phi)$ is a local conservation law.

4.5 Application to energy-momentum tensors

We present here an application for Theorem 4.12 to covariant divergence-free energy-momentum tensors.

Corollary 4.15 – Local conservation laws for divergence-free contravariant 2-tensors Let $(\mathcal{M}^m, g)$ be a real analytic m-dimensional Riemannian manifold, ∇ be the Levi-Civita connection and T be a contravariant 2-tensor with a vanishing covariant divergence. Then there exists a conservation law for T on $\mathcal{M} \times \mathbb{R}^{m+(m-1)^2}$.

Proof. Consider a contravariant 2-tensor $\mathrm{T} \in \Gamma(\mathrm{T}\mathcal{M} \otimes \mathrm{T}\mathcal{M})$, expressed locally $\mathrm{T} = \mathrm{T}^{\lambda\mu}\xi_\lambda \otimes \xi_\mu$, where $(\xi_1, \ldots, \xi_m)$ is moving frame dual to the moving coframe $(\eta^1, \ldots, \eta^m)$. The volume form is denoted by $\eta^\Lambda = \eta^1 \wedge \cdots \wedge \eta^m$. Using the interior product, we can associate any bivector T with a $\mathrm{T}\mathcal{M}$-valued $(m-1)$-differential form τ defined as follows:

$$\begin{aligned}\Gamma(\mathrm{T}\mathcal{M} \otimes \mathrm{T}\mathcal{M}) &\longrightarrow \Gamma(\mathrm{T}\mathcal{M} \otimes \wedge^{(m-1)}\mathrm{T}^*\mathcal{M})\\ \mathrm{T} = \mathrm{T}^{\lambda\mu}\xi_\lambda \otimes \xi_\mu &\longmapsto \tau = \xi_\lambda \otimes \tau^\lambda = \xi_\lambda \otimes \left(\mathrm{T}^{\lambda\mu}\xi_\mu \lrcorner \eta^\Lambda\right)\end{aligned}$$

Lemma 4.16 – Energy-momentum tensor vs $\mathrm{T}\mathcal{M}$-valued differential form Let $(\mathcal{M}^m, g)$ be an m-dimensional Riemannian manifold. Let T be a twice contravariant tensor and let τ be the associated tangent bundle-valued differential $(m-1)$-form. Then τ is covariantly closed if and only if the tensor is covariant divergence-free.

Using lemma 4.16, we conclude that for an m-dimensional Riemannian manifold $\mathcal{M}$, theorem 4.12, applied to $(\mathcal{M}^m, g)$ relatively to τ, i.e., for the ingredients $(\mathrm{T}\mathcal{M}, \mathcal{M}, g, \nabla, \tau)_{(m-1)}$, assures the existence of a (generalized) isometric embedding $\Psi : \mathrm{T}\mathcal{M} \longrightarrow \mathcal{M} \times \mathbb{R}^{m+(m-1)^2}$ such that $\mathrm{d}(\Psi(\tau)) = 0$ is a conservation law for a covariant divergence-free energy-momentum tensor T. For instance, if $\dim\mathcal{M} = 4$, then $\Psi(\tau)$ is a closed differential 3-form on $\mathcal{M}$ with values in $\mathbb{R}^{13}$. $\square$

PROOF OF LEMMA 4.16

The tangent space $\mathrm{T}\mathcal{M}$ is endowed with the Levi-Civita connection ∇. The covariant derivative of τ is

$$\mathrm{d}_\nabla \tau = \xi_\lambda \otimes (\mathrm{d}\tau^\lambda + \eta^\lambda_\mu \wedge \tau^\mu). \tag{4.11}$$

On one hand, by using Cartan's first-structure equation that expresses the vanishing of the torsion of the Levi-Civita connection and the expression of the Christoffel symbols in terms of the connection 1-form, we obtain

$$\mathrm{d}\tau^\lambda = \mathrm{d}\Big(\mathrm{T}^{\lambda\mu}(\xi_\mu \lrcorner \eta^\Lambda)\Big) = \mathrm{d}(\mathrm{T}^{\lambda\mu}) \wedge (\xi_\mu \lrcorner \eta^\Lambda) + \mathrm{T}^{\lambda\mu}\mathrm{d}(\xi_\mu \lrcorner \eta^\Lambda) = \Big(\xi_\mu(\mathrm{T}^{\lambda\mu}) + \mathrm{T}^{\lambda\mu}\Gamma^\nu_{\nu\mu}\Big)\eta^\Lambda \tag{4.12}$$

and

$$\eta^\lambda_\mu \wedge \tau^\mu = \eta^\lambda_\mu \wedge \mathrm{T}_{\mu\nu}(\xi_\nu \lrcorner \eta^\Lambda) = \Big(\mathrm{T}^{\mu\nu}\Gamma^\lambda_{\nu\mu}\Big)\eta^\Lambda \tag{4.13}$$

are obtained by using the following partial computations

$$\begin{aligned} \mathrm{d}(\mathrm{T}^{\lambda\mu}) &= \xi_\nu(\mathrm{T}^{\lambda\mu})\eta^\nu && (4.14)\\ \xi_\mu \lrcorner \eta^\Lambda &= (-1)^{\mu-1}\eta^{\Lambda\setminus\mu} = (-1)^{\mu-1}\eta^1 \wedge \cdots \wedge \eta^{\mu-1} \wedge \eta^{\mu+1} \wedge \cdots \wedge \eta^m && (4.15)\\ \mathrm{d}(\xi_\mu \lrcorner \eta^\Lambda) &= \Gamma^\nu_{\nu\mu}\eta^\Lambda && (4.16) \end{aligned}$$

The covariant derivative of τ is finally

$$\mathrm{d}_\nabla \tau = \xi_\lambda \otimes \Big[\Big(\xi_\mu(\mathrm{T}^{\lambda\mu}) + \mathrm{T}^{\lambda\mu}\Gamma^\nu_{\nu\mu} + \mathrm{T}^{\mu\nu}\Gamma^\lambda_{\nu\mu}\Big)\eta^\Lambda\Big]. \tag{4.17}$$

On the other hand, a straightforward computation of the divergence of the bivector leads to

$$\nabla_\mu \mathrm{T}^{\lambda\mu} = \xi_\mu(\mathrm{T}^{\lambda\mu}) + \mathrm{T}^{\lambda\mu}\Gamma^\nu_{\nu\mu} + \mathrm{T}^{\mu\nu}\Gamma^\lambda_{\nu\mu} \quad \text{for all } \lambda = 1, \ldots, m. \tag{4.18}$$

We then conclude that (see the subappendix below for a detailed computation in the case of a surface):

$$\mathrm{d}_\nabla \tau = 0 \Leftrightarrow \nabla_\mu \mathrm{T}^{\lambda\mu} = 0 \quad \forall \lambda = 1, \ldots, m. \tag{4.19}$$

4.A DETAILED PROOF OF LEMMA 4.16 FOR SURFACES

Some details of the computations of the proof of lemma 4.16 are not presented above. In order to help the reader understand them, the detailed computations of the covariant derivative of the tangent bundle valued differential 1-form for a surface are presented in this subappendix.

Let $(\mathcal{M}^2, g)$ be a real analytic Riemannian surface. As in the previous section, (η^1, η^2) is an orthonormal coframe, and denote by (ξ_1, ξ_2) the associated orthonormal frame. The volume form $\mathrm{vol}_{\mathcal{M}^2}$ is $\eta^1 \wedge \eta^2$. Let T be a twice contravariant tensor. T is then expressed in the moving frame as follows:

$$\mathrm{T} = \mathrm{T}^{\lambda\mu}\xi_\lambda \otimes \xi_\mu = \mathrm{T}^{11}\xi_1 \otimes \xi_1 + \mathrm{T}^{12}\xi_1 \otimes \xi_2 + \mathrm{T}^{21}\xi_2 \otimes \xi_1 + \mathrm{T}^{22}\xi_2 \otimes \xi_2. \tag{4.20}$$

Then the associated $\mathrm{T}\mathcal{M}^2$-valued differential 1-form τ is:

$$\begin{aligned}
\tau := \xi_\lambda \otimes \tau^\lambda &= \xi_\lambda \otimes \Big(\mathrm{T}^{\lambda\mu}(\xi_\mu \lrcorner \eta^\Lambda)\Big) = \xi_\lambda \otimes \Big(\mathrm{T}^{\lambda\mu}(\xi_\mu \lrcorner \eta^1 \wedge \eta^2)\Big) \\
&= \xi_1 \otimes \Big(\mathrm{T}^{11}(\xi_1 \lrcorner \eta^1 \wedge \eta^2) + \mathrm{T}^{12}(\xi_2 \lrcorner \eta^1 \wedge \eta^2)\Big) + \xi_2 \otimes \Big(\mathrm{T}^{21}(\xi_1 \lrcorner \eta^1 \wedge \eta^2) + \mathrm{T}^{22}(\xi_2 \lrcorner \eta^1 \wedge \eta^2)\Big) \\
&= \xi_1 \otimes \underbrace{(\mathrm{T}^{11}\eta^2 - \mathrm{T}^{12}\eta^1)}_{:=\tau^1} + \xi_2 \otimes \underbrace{(\mathrm{T}^{21}\eta^2 - \mathrm{T}^{22}\eta^1)}_{:=\tau^2}
\end{aligned} \tag{4.21}$$

The covariant derivative of τ is :

$$\mathrm{d}_\nabla \tau = \xi_1 \otimes (\mathrm{d}\tau^1 + \eta^1_2 \wedge \tau^2) + \xi_2 \otimes (\mathrm{d}\tau^2 + \eta^2_1 \wedge \tau^1) \tag{4.22}$$

for $\lambda = 1$ and 2, and by using Cartan's first-structure equation, the expression (1.18) of the Christoffel symbols in term of the connection 1-form of ∇, and their symmetries (1.19), we obtain:

$$\begin{aligned}
\mathrm{d}\tau^\lambda &= \mathrm{d}\Big(\mathrm{T}^{\lambda 1}\eta^2 - \mathrm{T}^{\lambda 2}\eta^1\Big) = \mathrm{d}(\mathrm{T}^{\lambda 1}) \wedge \eta^2 + \mathrm{T}^{\lambda 1}\mathrm{d}\eta^2 - \mathrm{d}(\mathrm{T}^{\lambda 2}) \wedge \eta^1 - \mathrm{T}^{\lambda 2}\mathrm{d}\eta^1 \\
&= \xi_\mu(\mathrm{T}^{\lambda 1})\eta^\mu \wedge \eta^2 - \mathrm{T}^{\lambda 1}\eta^2_1 \wedge \eta^1 - \xi_\mu(\mathrm{T}^{\lambda 2})\eta^\mu \wedge \eta^1 + \mathrm{T}^{\lambda 2}\eta^1_2 \wedge \eta^2 \\
&= \xi_1(\mathrm{T}^{\lambda 1})\eta^1 \wedge \eta^2 + \mathrm{T}^{\lambda 1}\eta^1 \wedge \eta^2_1 + \xi_2(\mathrm{T}^{\lambda 2})\eta^1 \wedge \eta^2 + \mathrm{T}^{\lambda 2}\eta^1_2 \wedge \eta^2 \\
&= \xi_1(\mathrm{T}^{\lambda 1})\eta^1 \wedge \eta^2 + \mathrm{T}^{\lambda 1}\eta^1 \wedge \Gamma^2_{\mu 1}\eta^\mu + \xi_2(\mathrm{T}^{\lambda 2})\eta^1 \wedge \eta^2 + \mathrm{T}^{\lambda 2}\Gamma^1_{\mu 2}\eta^\mu \wedge \eta^2 \\
&= \xi_1(\mathrm{T}^{\lambda 1})\eta^1 \wedge \eta^2 + \mathrm{T}^{\lambda 1}\Gamma^2_{21}\eta^1 \wedge \eta^2 + \xi_2(\mathrm{T}^{\lambda 2})\eta^1 \wedge \eta^2 + \mathrm{T}^{\lambda 2}\Gamma^1_{12}\eta^1 \wedge \eta^2 \\
&= \Big(\xi_1(\mathrm{T}^{\lambda 1}) + \xi_2(\mathrm{T}^{\lambda 2}) + \mathrm{T}^{\lambda 1}\Gamma^2_{21} + \mathrm{T}^{\lambda 2}\Gamma^1_{12}\Big)\eta^1 \wedge \eta^2 = \Big(\xi_\mu(\mathrm{T}^{\lambda\mu}) + \mathrm{T}^{\lambda\mu}\Gamma^\nu_{\nu\mu}\Big)\eta^1 \wedge \eta^2
\end{aligned} \tag{4.23}$$

and

$$\eta^1_2 \wedge \tau^2 = \Gamma^1_{\mu 2}\eta^\mu \wedge (\mathrm{T}^{21}\eta^2 - \mathrm{T}^{22}\eta^1) = \Big(\Gamma^1_{12}\mathrm{T}^{21} + \Gamma^1_{22}\mathrm{T}^{22}\Big)\eta^1 \wedge \eta^2 \tag{4.24}$$

$$\eta^2_1 \wedge \tau^1 = \Gamma^2_{\mu 1}\eta^\mu \wedge (\mathrm{T}^{11}\eta^2 - \mathrm{T}^{12}\eta^1) = \Big(\Gamma^2_{12}\mathrm{T}^{11} + \Gamma^1_{22}\mathrm{T}^{12}\Big)\eta^1 \wedge \eta^2 \tag{4.25}$$

Therefore,

$$\mathrm{d}_\nabla \tau = \Big[\xi_1 \otimes \Big(\xi_\mu(\mathrm{T}^{1\mu}) + \mathrm{T}^{1\mu}\Gamma^\nu_{\nu\mu} + \mathrm{T}^{\mu\nu}\Gamma^1_{\nu\mu}\Big) + \xi_2 \otimes \Big(\xi_\mu(\mathrm{T}^{2\mu}) + \mathrm{T}^{2\mu}\Gamma^\nu_{\nu\mu} + \mathrm{T}^{\mu\nu}\Gamma^2_{\nu\mu}\Big)\Big]\eta^1 \wedge \eta^2. \tag{4.26}$$

Chapter 5

A general strategy and the conservation law case

As expressed by the title, this chapter is dedicated to both presenting a solving strategy for the generalized isometric embedding problem and establishing the solution to the generalized isometric embedding problem in the conservation laws case, i.e., $p = m - 1$. In section 1, we investigate the problem locally and express the problem in terms of differential forms. The generalized isometric embedding problem turns out to be equivalent to looking for integral manifolds for an exterior differential system, and generalized notions are defined in this process. In section 2, we specialize in the conservation laws case, and then give the answer to the generalized isometric problem. The key for this result is the lemma 5.14. Finally, by considering the case $(\wedge^2 \mathrm{T}\mathcal{M}^3, \mathcal{M}^3, g, \mathrm{Id}_{\mathrm{T}\mathcal{M}^3})$ in the subappendix of this chapter, we illustrate the several definitions, equations, notations and explanations of the previous two sections.

5.1 The generalized isometric embedding problem via EDS

The generalized isometric embedding ingredients are the data $(\mathbb{V}^n, \mathcal{M}^m, g, \nabla, \phi)_p$, i.e, a vector bundle $\mathbb{V}^n$ of rank n over an m-dimensional manifold $\mathcal{M}^m$, a metric bundle g, a g-compatible connection and a covariantly closed $\mathbb{V}$-valued differential p-form ϕ. As a special case, the isometric embedding ingredients, which are provided by a Riemannian manifolds $\mathcal{M}^m, g$, are $(\mathrm{T}\mathcal{M}, \mathcal{M}, g, \nabla, \mathrm{Id}_{\mathrm{T}\mathcal{M}})_1$. Recall that the condition for $\mathrm{Id}_{\mathrm{T}\mathcal{M}}$ to be covariantly constant is equivalent, by the proposition 4.6, to the fact that the connection ∇ is torsion-free.

The generalized isometric embedding problem can be represented by the following diagram, where $\mathrm{N}^n_{m,p}$ is an integer that has to be defined in terms of the problem's data: n, m and p.

$$\begin{tikzcd}
g, \nabla, \quad \mathbb{V}^n \arrow[r, hook, "\Psi"] \arrow[d, "(\mathrm{d}_\nabla \phi)_p = 0"'] & \mathcal{M}^m \times \mathbb{R}^{\mathrm{N}^n_{m,p}} \arrow[d, "\mathrm{d}\Psi(\phi) = 0"] \\
\mathcal{M}^m & \mathcal{M}^m
\end{tikzcd}$$

Figure 5.1: Generalized isometric embedding

Denote by $\kappa^n_{m,p}$ the *embedding codimension*, i.e., roughly speaking, in how many dimensions one should extend the fiber in order to be able to achieve the desired embedding, and hence,

$\mathrm{N}^n_{m,p} = n + \kappa^n_{m,p}$. We also adopt the Einstein summation convention, i.e., there is a summation when the same index is repeated in high and low positions. However, we will write the sign $\sum$ and make explicit the values of the summation indices where necessary. Since the rank of the fiber and the manifold's dimension may be different, and since the dimension of the target embedding space is larger than the fiber, we thus adopt the following convention on the indices:

Notation 5.1 — Index conventions

- $\lambda, \mu, \nu = 1, \dots, m$ are the manifold indices ($\mathcal{M}^m$).
- $i, j, k = 1, \dots, n$ are the fiber indices ($\mathbb{V}^n$).
- $\mathrm{A}, \mathrm{B}, \mathrm{C} = 1, \dots, n + \kappa^n_{m,p}$ are the total embedding indices ($\mathbb{R}^{\mathrm{N}^n_{m,p}}$).
- $a, b, c = n+1, \dots, n + \kappa^n_{m,p}$ are the extension indices.

In the following, we will fix a moving coframe $\mathcal{M}$ denoted by $\eta = (\eta^1, \dots, \eta^m)$ and will fix a g-orthonormal moving frame of $\mathbb{V}$ denoted by $\mathrm{E} = (\mathrm{E}_1, \dots, \mathrm{E}_n)$. Thus a $\mathbb{V}$-valued differential form $\phi \in \Gamma(\mathbb{V} \otimes \wedge \mathrm{T}^*\mathcal{M}))$ can be, locally, expressed as follows:

$$\phi = \mathrm{E}_i \phi^i = \mathrm{E}_i \psi^i_{\lambda_1, \dots, \lambda_p} \eta^{\lambda_1, \dots, \lambda_p} \tag{5.1}$$

where $\psi^i_{\lambda_1, \dots, \lambda_p}$ are functions on $\mathcal{M}$. **We assume that $\mathbf{1} \leqslant \lambda_1 < \cdots < \lambda_p \leqslant \mathbf{m}$ in the summation and $\eta^{\lambda_1, \dots, \lambda_p}$ means $\eta^{\lambda_1} \wedge \cdots \wedge \eta^{\lambda_p}$.**

Example 5.2 — The isometric embedding case. Let $(\mathcal{M}, g)$ be an m-dimensional Riemannian manifold and consider the isometric embedding ingredients $(\mathrm{T}\mathcal{M}, \mathcal{M}, g, \nabla, \mathrm{Id}_{\mathrm{T}\mathcal{M}})$. Then $\phi = \mathrm{d}_{\mathrm{T}\mathcal{M}}$ can be viewed as a $\mathrm{T}\mathcal{M}$-valued differential 1-form and thus is expressed in a g-orthonormal coframe $(\eta^1, \dots, \eta^m)$ dual to $(\mathrm{E}_1, \dots, \mathrm{E}_m)$ as follows:

$$\mathrm{Id}_{\mathrm{T}\mathcal{M}} = \mathrm{E}_i \phi^i = \mathrm{E}_i \psi^i_j \eta^j = \mathrm{E}_\lambda \eta^\lambda, \tag{5.2}$$

where the functions ψ^i_λ are the Kronecker symbols δ^i_λ.

Definition 5.3 — Generalized torsion Let $\mathbb{V}^n$ be a vector bundle of rank n over an m-dimensional manifold $\mathcal{M}^m$, g a metric bundle on $\mathbb{V}^n$, ∇ a g-connection on $\mathbb{V}^n$, and ϕ a $\mathbb{V}^n$-valued differential p-form. In other words, we consider the generalized isometric embedding problem's ingredients except that ϕ is not required to be covariantly closed. Then, the generalized torsion of the connection ∇ relative to the $\mathbb{V}^n$-valued differential p-form ϕ, or for short, the ϕ-torsion, is the $\mathbb{V}$-valued differential $(p+1)$-form defined by $\Theta = (\Theta^i) := \mathrm{d}_\nabla \phi$, i.e., in a local frame:

$$\Theta = \mathrm{E}_i \Theta^i := \mathrm{E}_i (\mathrm{d}\phi^i + \eta^i_j \wedge \phi^j) \tag{5.3}$$

where (η^i_j) is the connection 1-form of ∇ which is an $\mathfrak{o}(n)$-valued differential 1-form[1]. Moreover, if Θ vanishes, the connection is said to be ϕ-torsion-free.

Example 5.4 — The "usual" torsion of a connection. In the isometric embedding case, the generalized torsion is nothing but the usual torsion, i.e., a $\mathrm{T}\mathcal{M}$-valued differential 2-form as defined in 1.13.

The same relationship between $\mathrm{Id}_{\mathrm{T}\mathcal{M}}$ and the torsion-free condition of the connection in the isometric embedding case exists in the generalized isometric embedding case, as shown by the following proposition.

[1]The connection ∇ is compatible with the metric bundle g.

Proposition 5.5 Let $(\mathbb{V}^n, \mathcal{M}, g, \nabla, \phi)_p$ be the generalized isometric ingredients. The condition of being covariantly closed for ϕ is equivalent to the fact that the connection ∇ is ϕ-torsion-free.

Proof. $\mathrm{d}_\nabla \phi = \mathrm{d}_\nabla(\mathrm{E}_i \phi^i) = \nabla(\mathrm{E}_i) \wedge \phi^i + \mathrm{E}_i \mathrm{d}\phi^i = \mathrm{E}_j \eta_i^j \wedge \phi^i + \mathrm{E}_i \mathrm{d}\phi^i = \mathrm{E}_i(\mathrm{d}\phi^i + \eta_j^i \wedge \phi^j) = 0.$ □

Let us now formulate the generalized isometric embedding problem equations by means of exterior differential systems. Let ω be the connection 1-form of the standard connection on $\mathbb{R}^{\mathrm{N}^n_{m,p}}$. For convenience, instead of working with orthonormal frames on $\mathbb{R}^{\mathrm{N}^n_{m,p}}$ in order to express the connection 1-form ω, we choose an adapted geometry. The reason is the same as when using adapted frames for the isometric embedding of surfaces in a three dimensional Euclidean space.

Consider the flat connection 1-form ω on the Stiefel space $\mathrm{SO}(n+\kappa^n_{m,p})/\mathrm{SO}(\kappa^n_{m,p})$, the n-adapted frames of $\mathbb{R}^{(n+\kappa^n_{m,p})}$, i.e., the set of orthonormal families of n vectors $(e_1, \ldots, e_n)$ of $\mathbb{R}^{(n+\kappa^n_{m,p})}$ which can be completed by orthonormal $\kappa^n_{m,p}$ vectors $(e_{n+1}, \ldots, e_{n+\kappa^n_{m,p}})$ to obtain an orthonormal set of $(n+\kappa^n_{m,p})$ vectors. Denote by Υ such a class of coframe. Since we work locally, we will assume without loss of generality that we are given a cross-section $(e_{n+1}, \ldots, e_{n+\kappa^n_{m,p}})$ of the bundle fibration $\mathrm{SO}(n+\kappa^n_{m,p}) \longrightarrow \mathrm{SO}(n+\kappa^n_{m,p})/\mathrm{SO}(\kappa^n_{m,p})$. The flat standard 1-form of the connection ω is defined as follows:

$$\omega_j^i = \langle e_i, \mathrm{d}e_j \rangle_{\mathbb{R}^{\mathrm{N}^n_{m,p}}} \quad \text{and} \quad \omega_i^a = \langle e_a, \mathrm{d}e_i \rangle_{\mathbb{R}^{\mathrm{N}^n_{m,p}}} \tag{5.4}$$

where $\langle , \rangle_{\mathbb{R}^{\mathrm{N}^n_{m,p}}}$ is the standard inner product on $\mathbb{R}^{n+\kappa^n_{m,p}}$. Notice that ω satisfies Cartan's structure equations (1.15), and since the connection is flat, the curvature of ω vanishes.

An isometry between two Riemannian manifolds maps an orthonormal set of vectors of the source manifold to an orthonormal set of vectors of the target manifold. The conservation law Ψ, if it exists, maps then $(\mathrm{E}_1, \ldots, \mathrm{E}_n)$ to an element of the Stiefel space. Let us assume for instance that such a map Ψ exists, then if $e_i = \Psi(\mathrm{E}_i)$, the condition $\mathrm{d}\Psi(\phi) = 0$ yields to

$$e_i(\mathrm{d}\phi^i + \omega_j^i \wedge \phi^j) + e_a(\omega_i^a \wedge \phi^i) = 0, \tag{5.5}$$

a condition which is satisfied if and only if

$$\eta_j^i = \Psi^*(\omega_j^i) \quad \text{and} \quad \Psi^*(\omega_i^a) \wedge \phi^i = 0. \tag{5.6}$$

Solutions to the generalized isometric embedding problem is equivalent then in finding moving frames $(e_1, \ldots e_n, e_{n+1}, \ldots, e_{n+\kappa^n_{m,p}})$ in the Stiefel space such that there exist m-dimensional integral manifolds of the exterior ideal generated by the naive exterior differential system

$$\{\omega_j^i - \eta_j^i, \omega_i^a \wedge \phi^i\}_{\mathrm{alg}} \text{ on the product manifold } \mathbf{\Sigma}^n_{m,p} = \mathcal{M} \times \frac{\mathrm{SO}(n+\kappa^n_{m,p})}{\mathrm{SO}(\kappa^n_{m,p})}. \tag{5.7}$$

Strictly speaking, differential forms live in different spaces. Indeed, one should consider the projections $\pi_{\mathcal{M}}$ and π_{St} of $\mathbf{\Sigma}^n_{m,p}$ on $\mathcal{M}$ and the Stiefel space, and consider the ideal on $\mathbf{\Sigma}^n_{m,p}$ generated by $\pi^*_{\mathcal{M}}(\eta_j^i) - \pi^*_{\mathrm{St}}(\omega_j^i)$ and $\pi^*_{\mathrm{St}}(\omega_i^a) \wedge \pi^*_{\mathcal{M}}(\phi^i)$. It seems reasonable however to simply write $\{\omega_j^i - \eta_j^i, \omega_i^a \wedge \phi^i\}_{\mathrm{alg}}$.

To find integral manifolds of the naive EDS, we would need to check that the exterior ideal is closed under the differentiation. However, this turns out not to be the case. The idea is then to add to the naive EDS the exterior differential of the forms that generate it and as a consequence, we obtain a closed one.

Notice that some objects that we are dealing with in the following have a geometric meaning in the tangent bundle case with a standard 1-form ($\phi = \mathrm{Id}_{\mathrm{T}\mathcal{M}}$) but not in an arbitrary vector bundle case, as we noticed earlier with the notion of torsion of a connection. That leads us to extend these notions in a generalized sense in such a way that we recover the standard notions in the tangent bundle case. First of all, the Cartan lemma, which in the isometric embedding problem implies the symmetry of the second fundamental form, does not hold. Consequently, we cannot assure nor assume that the coefficients of the second fundamental form are symmetric as in the isometric embedding problem. In fact, we will show that these conditions should be replaced by *generalized Cartan identities* that express how coefficients of the second fundamental form are related to each other, and of course, we recover the usual symmetry in the tangent bundle case. Another difficulty is the analogue of the Bianchi identity of the curvature tensor. We will define *generalized Bianchi identities* relative to the covariantly closed vector bundle valued differential p-form and a *generalized curvature tensor space* which correspond respectively, in the tangent bundle case, to the usual Bianchi identities and the Riemann curvature tensor space. Finally, besides the *generalized Cartan identities* and *generalized curvature tensor space*, we will make use of a *generalized Gauss map*.

The key to the proof of Theorem 4.12 is Lemma 5.14 for two main reasons: on the one hand, it assures the existence of suitable coefficients of the second fundamental form that satisfy the *generalized Cartan identities* and the *generalized Gauss equation*, properties that simplify the computation of the Cartan characters. On the other hand, the lemma gives the minimal required *embedding codimension* $\kappa^n_{m,m-1}$ that ensures the desired embedding. Using Lemma 5.14, we give another proof of Theorem 4.12 by an explicit construction of an ordinary integral flag. When the existence of an integral manifold is established, we just need to project it on $\mathcal{M} \times \mathbb{R}^{n+\kappa^n_{m,p}}$.

Proposition-Definition 5.6 — The generalized isometric embedding EDS The closure of the naive EDS $\{\omega^i_j - \eta^i_j, \omega^a_i \wedge \phi^i\}$ on the manifold $\boldsymbol{\Sigma}^n_{m,p}$ is the EDS $\mathcal{I}^n_{m,p}$, called the generalized isometric embedding EDS, and is defined as follows:

$$\mathcal{I}^n_{m,p} = \{\omega^i_j - \eta^i_j, \omega^i_a \wedge \omega^a_j + \Omega^i_j, \omega^a_i \wedge \phi^i\}_{\mathrm{alg}} \tag{5.8}$$

where $\Omega^i_j = \mathrm{d}\eta^i_j + \eta^i_k \wedge \eta^k_j$.

Note that the decomposition of the curvature 2-form Ω^i_j in the moving frame is

$$\Omega^i_j = \frac{1}{2}\mathcal{R}^i_{j;\lambda\mu}\eta^\lambda \wedge \eta^\mu. \tag{5.9}$$

Proof. The generalized torsion-free of the connection implies that $\mathrm{d}(\omega^a_i \wedge \phi^i) \equiv 0$ modulo the naive EDS, but Cartan's second-structure equation yields to $\mathrm{d}(\omega^i_j - \eta^i_j) \equiv \omega^i_a \wedge \omega^a_j + \Omega^i_j$ modulo the naive EDS, where $\Omega = (\Omega^i_j)$ is the curvature 2-form of the connection. □

A first covariant derivative of ϕ has led to the generalized torsion. A second covariant derivative of ϕ gives rise to *generalized Bianchi identities* defined as follows:

Definition 5.7 — Generalized Bianchi identities Let $(\mathbb{V}^n, \mathcal{M}^m, g, \nabla, \phi)_p$ be the generalized isometric embedding problem's ingredients. The curvature tensor then satisfies the generalized

Bianchi identities, encoded by the vanishing of a $\mathbb{V}$-valued differential $(p+2)$-form $\mathcal{B}^n_{m,p}$, defined as follows:

$$\mathcal{B}^n_{m,p} := \mathrm{E}_i(\Omega^i_j \wedge \phi^j) = 0. \tag{5.10}$$

The generalized Bianchi identities are nothing but a condense way of writing a system of equations that the curvature tensors of the connection ∇ must satisfy. Therefore, 5.10 is equivalent to the following system:

$$\sum_{1\leqslant\lambda<\mu\leqslant n} \mathcal{R}^i_{j;\lambda\mu}\eta^\lambda \wedge \eta^\mu \wedge \phi^j = 0 \quad \forall i = 1, \ldots, n. \tag{5.11}$$

Thus, it is then natural to consider the space of curvature tensors that satisfy the generalized Bianchi identities when dealing with theses geometries.

Definition 5.8 – Generalized curvature space Let $(\mathbb{V}^n, \mathcal{M}^m, g, \nabla, \phi)_p$ be the generalized isometric embedding problem's ingredients. Then the generalized curvature tensor space defined at some point is

$$\mathcal{K}^n_{m,p} := \{(\mathcal{R}^i_{j;\lambda\mu}) \in \wedge^2(\mathbb{R}^n) \otimes \wedge^2(\mathbb{R}^m) |\, \Omega^i_j \wedge \phi^j = 0, \forall i = 1, \ldots, n\}. \tag{5.12}$$

Examples 5.9 – Generalized curvature spaces.

1. **Covariantly closed vector bundle valued form of codegree one**: The generalized Bianchi identities are trivially satisfied for covariantly closed vector bundle valued differential forms of codegree one, because a differential $(m+1)$-form on an m-dimensional manifold is zero. Consequently, $\mathcal{K}^n_{m,m-1} = \wedge^2(\mathbb{R}^n) \otimes \wedge^2(\mathbb{R}^m)$.

2. **Riemann curvature tensor space**: In the isometric embedding case, the generalized Bianchi identities are the first Bianchi identities, and hence, the generalized curvature tensor $\mathcal{K}^m_{m,1} = \{(\mathcal{R}^i_{j;\lambda\mu}) \in \mathrm{S}^2(\wedge^2(\mathbb{R}^m)) |\, \mathcal{R}^i_{j;kl} + \mathcal{R}^i_{l;jk} + \mathcal{R}^i_{k;lj} = 0\}$.

All of the data is analytic and we can then apply the Cartan–Kähler theory if we are able to check the involution property of the exterior differential system by constructing an m-integral flag. If the exterior ideal $\mathcal{I}^n_{m,p}$ satisfies the Cartan test, the flag is then ordinary and by the Cartan–Kähler theorem, there exist integral manifolds of $\mathcal{I}^n_{m,p}$. To be able to project the product manifold $\boldsymbol{\Sigma}^n_{m,p}$ on $\mathcal{M}$, we also need to show the existence of m-dimensional integral manifolds on which the volume form on $\eta^{1,\ldots,m}$ on $\mathcal{M}$ does not vanish.

The EDS is not involutive and hence we ”prolong” by introducing the new variables $\mathrm{H}^a_{i\lambda}$. Let us express the 1-forms ω^a_i in the coframe $(\eta^1, \ldots, \eta^m)$ in order to later compute the Cartan characters. Let $\mathcal{W}^n_{m,p}$ be an $\kappa^n_{m,p}$-dimensional Euclidean space, which is a model of a normal space for the embedding. We then write

$$\omega^a_i = \mathrm{H}^a_{i\lambda}\eta^\lambda \quad \text{where} \quad \mathrm{H}^a_{i\lambda} \in \mathcal{W}^n_{m,m-1} \otimes \mathbb{R}^n \otimes \mathbb{R}^m \tag{5.13}$$

and define the forms $\pi^a_i = \omega^a_i - \mathrm{H}^a_{i\lambda}\eta^\lambda$. The $\mathrm{H}^a_{i\lambda}$ can be seen as coefficients of the second fundamental form. We can also consider $\mathrm{H}_{i\lambda} = (\mathrm{H}^a_{i\lambda})$ as a vector of $\mathcal{W}^n_{m,p}$. The forms that generate algebraically $\mathcal{I}^n_{m,p}$ are then expressed as follows:

$$\sum_a \omega^a_i \wedge \omega^a_j - \Omega^i_j = \sum_a \pi^a_i \wedge \pi^a_j + \sum_a (\mathrm{H}^a_{j\lambda}\pi^a_i - \mathrm{H}^a_{i\lambda}\pi^a_j) \wedge \eta^\lambda + \frac{1}{2}\underbrace{(\mathrm{H}_{i\lambda}.\mathrm{H}_{j\mu} - \mathrm{H}_{i\mu}.\mathrm{H}_{j\lambda} - \mathcal{R}^i_{j;\lambda\mu})}_{*}\eta^\lambda \wedge \eta^\mu \tag{5.14}$$

$$\omega^a_i \wedge \phi^i = \psi^i_{\lambda_1\ldots\lambda_p}\pi^a_i \wedge \eta^{\lambda_1\ldots\lambda_p} + \sum_{\substack{\lambda=1,\ldots,m \\ 1\leqslant\mu_1<\cdots<\mu_p\leqslant m}} \overbrace{\mathrm{H}^a_{i\lambda}\psi^i_{\mu_1,\ldots,\mu_p}}^{**}\eta^{\lambda\mu_1\ldots\mu_p} \tag{5.15}$$

These new expressions of the forms in terms of vectors H and the differential 1-form π later lead us to compute the Cartan characters of an m-integral flag. The expressions marked with $(*)$ and $(**)$ are obstructions for the embedding. They must vanish in order to realize the generalized isometric embedding. Inspired by the isometric embedding case, we define:

Definition 5.10 — Generalized Gauss equation Let $(\mathbb{V}^n, \mathcal{M}^m, g, \nabla, \phi)_p$ be the generalized isometric embedding problem's ingredients. Let $\mathrm{H}^a_{i\lambda} \in \mathcal{W}^n_{m,p} \otimes \mathbb{R}^n \otimes \mathbb{R}^m$ be the coefficient of the second fundamental form, where $\mathcal{W}^n_{m,p}$ is a $\kappa^n_{m,p}$-Euclidean space. For $\mathcal{R}^i_{j;\lambda\mu}$ in $\mathcal{K}^n_{m,p}$, the generalized Gauss equation is:

$$\mathrm{H}_{i\lambda}.\mathrm{H}_{j\mu} - \mathrm{H}_{i\mu}.\mathrm{H}_{j\lambda} = \mathcal{R}^i_{j;\lambda\mu}. \tag{5.16}$$

Definition 5.11 — Generalized Cartan identities Let $(\mathbb{V}^n, \mathcal{M}^m, g, \nabla, \phi)_p$ be the generalized isometric embedding problem's ingredients. Let $\mathrm{H}^a_{i\lambda} \in \mathcal{W}^n_{m,p}\otimes\mathbb{R}^n\otimes\mathbb{R}^m$ be the coefficients of the second fundamental form, where $\mathcal{W}^n_{m,p}$ is a $\kappa^n_{m,p}$-Euclidean space. Let $\psi^i_{\mu_1\ldots\mu_p}$ be the coordinate of ϕ^i in the coframe η. The "symmetry condition" that the coefficients $\mathrm{H}^a_{i\lambda}$ should satisfy are:

$$\sum_{\substack{\lambda=1,\ldots,m \\ 1\leqslant\mu_1<\cdots<\mu_p\leqslant m}} \mathrm{H}^a_{i\lambda}\psi^i_{\mu_1,\ldots,\mu_p}\eta^{\lambda\mu_1\ldots\mu_p} = 0 \tag{5.17}$$

With the generalized Bianchi identities and the generalized Gauss equation, we define a mapping between the second fundamental form's coefficients and the generalized curvature space.

Definition 5.12 — Generalized Gauss map Let $(\mathbb{V}^n, \mathcal{M}^m, g, \nabla, \phi)_p$ be the ingredients of the generalized isometric embedding. Then the generalized Gauss map $\mathcal{G}^n_{m,p} : \mathcal{W}^n_{m,p}\otimes\mathbb{R}^n\otimes\mathbb{R}^m \longrightarrow \mathcal{K}^n_{m,p}$ is defined for $(\mathrm{H}^a_{i\lambda}) \in \mathcal{W}^n_{m,p}\otimes\mathbb{R}^n\otimes\mathbb{R}^m$ by

$$\Big(\mathcal{G}^n_{m,p}(\mathrm{H})\Big)^i_{j;\lambda\mu} = \sum_a (\mathrm{H}^a_{i\lambda}\mathrm{H}^a_{j\mu} - \mathrm{H}^a_{i\mu}\mathrm{H}^a_{j\lambda}). \tag{5.18}$$

The key is to determine the minimum embedding codimension $\kappa^n_{m,p}$ and to show the existence of such coefficients $\mathrm{H}^a_{i\lambda}$ that satisfy both the generalized Cartan identities and the generalized Gauss equations. Finally, to compute the characters C_λ, for $\lambda = 0,\ldots,m-1$, in order to check the involution by the Cartan test, we apply Proposition 2.34 to enumerate the number of linearly independent differential 1-forms $\sum(\mathrm{H}^a_{j\lambda}\pi^a_i - \mathrm{H}^a_{i\lambda}\pi^a_j)$ and $\psi^i_{\lambda_1\ldots\lambda_p}\pi^a_i$ that appear in equation 5.14 and 5.15.

5.2 SPECIALIZATION IN THE CONSERVATION LAW CASE

Let us specialize to the conservation laws case, i.e., the covariantly closed vector bundle valued form is of codegree one ($p = m-1$). The key is to construct an ordinary m-dimensional integral element of the generalized isometric embedding exterior ideal $\mathcal{I}^n_{m,m-1}$ on $\mathbf{\Sigma}^n_{m,m-1}$. Recall that the generalized Bianchi identities are trivial in this case.

Notation 5.13 We adopt these following notations for the conservation law case: $\Lambda = (1, 2, \ldots, m)$ and $\Lambda \smallsetminus \lambda = (1, \ldots, \lambda-1, \lambda+1, \ldots, m)$. We thus have $\eta^\Lambda = \eta^1 \wedge \cdots \wedge \eta^m$ and $\eta^{\Lambda\smallsetminus\lambda} = \eta^1 \wedge \cdots \wedge \eta^{\lambda-1} \wedge \eta^{\lambda+1} \wedge \cdots \wedge \eta^m$.

The different objects, equations and identities introduced above, become as follows in the conservation law case :

1. **The generalized isometric embedding EDS:** The generalized isometric embedding problem is equivalent to finding m-dimensional integral manifolds of $\mathcal{I}^n_{m,m-1} = \{\omega^i_j - \eta^i_j, \omega^i_a \wedge \omega^a_j + \Omega^i_j, \omega^a_i \wedge \phi^i\}_{\mathrm{alg}}$ on the product manifold $\mathbf{\Sigma}^n_{m,m-1}$.
2. **Generalized Bianchi identities:** They are no constraints of Bianchi type.
3. **Generalized curvature space:** The generalized curvature tensor space is $\mathcal{K}^n_{m,m-1} = \wedge^2\mathbb{R}^n \otimes \wedge^2\mathbb{R}^m$. Hence, in particular its dimension is $n(n-1)m(m-1)/4$.
4. **Generalized Gauss equation:** The generalized Gauss equation is $\mathrm{H}_{i\lambda}.\mathrm{H}_{j\mu} - \mathrm{H}_{i\mu}.\mathrm{H}_{j\lambda} = \mathcal{R}^i_{j;\lambda\mu}$, where $\mathrm{H}_{i\lambda}$ is viewed as a vector of the $\kappa^n_{m,m-1}$-dimensional Euclidean space $\mathcal{W}^n_{m,m-1}$.
5. **Generalized Cartan identities:** Generalized Cartan identities at M are
$$\sum_{\lambda=1,\ldots,m} (-1)^{\lambda+1}\mathrm{H}^a_{i\lambda}\psi^i_{\Lambda\smallsetminus\lambda} = 0 \quad \text{for all } a = n+1, , \ldots, n+\kappa^n_{m,m-1}, \tag{5.19}$$
assuming that ϕ is non-vanishing, and hence, the dimension of the coefficients space $\mathrm{H}^a_{i\lambda}$ that satisfy generalized Cartan identities is $(nm-1)\kappa^n_{m,m-1}$

The following lemma, a proof of which is given later, represents the key to the proof of Theorem 4.12.

Lemma 5.14 – The generalized Gauss map's submersitivity Let $\kappa^n_{m,m-1} \geqslant (m-1)(n-1)$. Let
$$\mathcal{H}^n_{m,m-1}(\mathrm{M}) \subset \mathcal{W}^n_{m,m-1} \otimes \mathbb{R}^n \otimes \mathbb{R}^m \tag{5.20}$$
be the open set consisting of those elements $\mathrm{H} = (\mathrm{H}^a_{i\lambda})$ so that the vectors $\{\mathrm{H}_{i\lambda} | i = 1, \ldots, n-1 \text{ and } \lambda = 1, \ldots, m-1\}$ are linearly independent as elements of $\mathcal{W}^n_{m,m-1}$ and satisfy the generalized Cartan identities. Then $\mathcal{G}^n_{m,m-1} : \mathcal{H}^n_{m,m-1} \longrightarrow \mathcal{K}^n_{m,m-1}$ is a surjective submersion.

Let $\mathcal{Z}^n_{m,m-1} = \{(\mathrm{M}, \Upsilon, \mathrm{H}) \in \mathbf{\Sigma}^n_{m,m-1} \times \mathcal{W}^n_{m,m-1} \otimes \mathbb{R}^n \otimes \mathbb{R}^m \,|\, \mathrm{H} \in \mathcal{H}^n_{m,m-1}(\mathrm{M})\}$. We conclude from Lemma 5.14 that $\mathcal{Z}^n_{m,m-1}$ is a submanifold[2] and hence,
$$\dim\, \mathcal{Z}^n_{m,m-1} = \dim\, \mathbf{\Sigma}^n_{m,m-1} + \dim\, \mathcal{H}^n_{m,m-1}(\mathrm{M}) \tag{5.21}$$

[2] $\mathcal{Z}^n_{m,m-1}$ is the fiber of $\mathcal{R}$ by a submersion. The surjectivity of $\mathcal{G}^n_{m,m-1}$ assures the non-emptiness.

where

$$\dim \mathbf{\Sigma}^n_{m,m-1} = m + \frac{n(n-1)}{2} + n\kappa^n_{m,m-1} \tag{5.22}$$

$$\dim \mathcal{H}^n_{m,m-1}(\mathrm{M}) = (nm-1)\kappa^n_{m,m-1} - \frac{n(n-1)m(m-1)}{4} \tag{5.23}$$

We define the map $\Phi^n_{m,m-1} : \mathcal{Z}^n_{m,m-1} \longrightarrow \mathcal{V}_m(\mathcal{I}^n_{m,m-1}, \eta^\Lambda)$ which associates $(\mathrm{M}, \Upsilon, \mathrm{H}) \in \mathcal{Z}^n_{m,m-1}$ with the m-plan on which the differential forms that generate algebraically $\mathcal{I}^n_{m,m-1}$ vanish and the volume form η^Λ on $\mathcal{M}$ does not vanish. $\Phi^n_{m,m-1}$ is then an embedding and hence $\dim \Phi(\mathcal{Z}^n_{m,m-1}) = \dim \mathcal{Z}^n_{m,m-1}$. In what follows, we prove that in fact $\Phi(\mathcal{Z}^n_{m,m-1})$ contains only ordinary m-integral elements of $\mathcal{I}^n_{m,m-1}$. Since the coefficients $\mathrm{H}^a_{i\lambda}$ satisfy the generalized Gauss equation and generalized Cartan identities, the differential forms that generate the exterior ideal $\mathcal{I}^n_{m,m-1}$ are as follows:

$$\omega^i_a \wedge \omega^a_j + \Omega^i_j = \sum_a \pi^a_i \wedge \pi^a_i + \sum_a (\mathrm{H}^a_{j\lambda}\pi^a_i - \mathrm{H}^a_{i\lambda}\pi^a_j)\eta^\lambda \tag{5.24}$$

$$\omega^a_i \wedge \phi^i = \psi^i_{\lambda_1\ldots\lambda_p}\pi^a_i \wedge \eta^{\lambda_1\ldots\lambda_p}. \tag{5.25}$$

The final step is then to compute the Cartan characters and to check by Cartan's test that $\Phi(\mathcal{Z}^n_{m,m-1})$ contains only ordinary m-integral flags. The Cartan–Kähler theorem then assures the existence of an m-integral manifold on which η^Λ does not vanish since the exterior ideal is in involution. We finally project the integral manifold on $\mathcal{M} \times \mathbb{R}^{n+\kappa}$. Let us notice that the requirement of the non-vanishing of the volume form η^Λ on the integral manifold yields to project the integral manifold on $\mathcal{M}$ and also to view it as a graph of a function f defined on $\mathcal{M}$ with values in the space of n-adapted orthonormal frames of $\mathbb{R}^{n+\kappa}$. In the isometric embedding problem, the composition of f with the projection of the frames on the Euclidean space is by construction the isometric embedding map.

5.2.1 Proof of lemma 5.14

The generalized Gauss map $\mathcal{G}^n_{m,m-1}$ defined on $\mathcal{W}^n_{m,m-1} \otimes \mathbb{R}^n \otimes \mathbb{R}^m$ with values in $\mathcal{K}^n_{m,m-1}$ is a submersion if and only if the differential $\mathrm{d}\mathcal{G}^n_{m,m-1} \in \mathcal{L}(\mathcal{W}^n_{m,m-1} \otimes \mathbb{R}^n \otimes \mathbb{R}^m; \mathcal{K}^n_{m,m-1})$, which has $m(m-1)n(n-1)/4$ lines and $\kappa^n_{m,m-1} \times m \times n$ columns, is of maximal rank.

In what follows, we make the assumption that $\psi^1_{\Lambda\smallsetminus m} = 1$ and $\psi^2_{\Lambda\smallsetminus m} = \cdots = \psi^n_{\Lambda\smallsetminus m} = 0$. It is always possible by changing the frame (E_i) and relabeling. With this assumption, the generalized Cartan identity shows that the vector $(\mathrm{H}^a_{1m})_a$ on a given point of the manifold, is a linear combination of the $\mathrm{H}_{i\lambda}$ where $\lambda \neq m$. When $n = m = 2$, we assume that the determinant $\det\psi = (\psi^1_1\psi^2_2 - \psi^1_2\psi^2_1) \neq 0$. In order to understand the proof of the submersitivity of $\mathcal{G}^n_{m,m-1}$, we explain and show the proof for two special cases: when the vector bundle is of rank 2 ($n = 2$) and when the manifold is a surface ($m = 2$). The proof of the surjectivity of the generalized Gauss map is established afterwards.

Submersitivity of the generalized Gauss map

We will proceed step by step in order to expound the proof of Lemma 5.14: For a warm-up, we start with the case $(\mathbb{V}^3, \mathcal{M}^2, g, \nabla, \phi)_1$, then the case of a general vector bundle over a surface, i.e., $(\mathbb{V}^n, \mathcal{M}^2, g, \nabla)_1$, next, the case of a vector bundle of rank 2 over an m-dimensional manifold, i.e., $(\mathbb{V}^2, \mathcal{M}^m, g, \nabla, \phi)_{m-1}$, and finally, we expound the conservation laws case, i.e., $(\mathbb{V}^n, \mathcal{M}^m, g, \nabla, \phi)_{m-1}$.

Recall that the generalized Gauss map associates $\mathrm{H} = (\mathrm{H}^a_{i\lambda})$ with $\left((\mathcal{G}^n_{m,m-1})^i_{j;\lambda\mu}\right) = (\mathrm{H}_{i\lambda}\mathrm{H}_{j\mu} - \mathrm{H}_{i\mu}\mathrm{H}_{j\lambda})^i_{j;\lambda\mu}$. The differential of $\mathcal{G}^n_{m,m-1}$ is then:

$$\mathrm{d}\mathcal{G}^n_{m,m-1} = \frac{\partial \mathcal{G}^n_{m,m-1}}{\partial \mathrm{H}^a_{i\lambda}} \mathrm{dH}^a_{i\lambda} \tag{5.26}$$

where

$$\mathrm{d}(\mathcal{G}^n_{m,m-1})^i_{j;\lambda\mu} = \mathrm{H}_{j\mu}\mathrm{dH}_{i\lambda} + \mathrm{H}_{i\lambda}\mathrm{dH}_{j\mu} - \mathrm{H}_{j\lambda}\mathrm{dH}_{i\mu} - \mathrm{H}_{i\mu}\mathrm{dH}_{j\lambda}. \tag{5.27}$$

Denote by $\varepsilon^i_{j;\lambda\mu}$ the natural basis on $\mathcal{K}^n_{m,m-1} = \wedge^2\mathbb{R}^n \otimes \wedge^2\mathbb{R}^m$.

THE CASE $(\mathbb{V}^3, \mathcal{M}^2, \mathbf{g}, \nabla, \phi)_1$: Consider a vector bundle $\mathbb{V}^3$ of rank 3 over a 2-dimensional differentiable manifold $\mathcal{M}^2$, endowed with a metric g and a connection ∇ compatible with g. Let ϕ be a non-vanishing covariantly closed $\mathbb{V}^2$-valued differential 1-form. By assumption,

$$\phi = \mathrm{E}_i\phi = \mathrm{E}_i\psi^i_\lambda\eta^\lambda = \begin{pmatrix} 1 & \psi^1_2 \\ 0 & \psi^2_2 \\ 0 & \psi^3_2 \end{pmatrix} \wedge \begin{pmatrix} \eta^1 \\ \eta^2 \end{pmatrix}. \tag{5.28}$$

The generalized Cartan identities 5.19 for each normal direction a are:

$$\mathrm{H}^a_{12} = \psi^1_2\mathrm{H}^a_{11} + \psi^2_2\mathrm{H}^a_{21} + \psi^3_2\mathrm{H}^a_{31}. \tag{5.29}$$

The curvature tensors' space is $\mathcal{K}^3_{2,1} = \wedge^2\mathbb{R}^3 \otimes \wedge^2\mathbb{R}^2 = \wedge^2\mathbb{R}^3 \otimes \mathbb{R} = \mathrm{span}\{\varepsilon^1_{2;12}, \varepsilon^1_{3;12}, \varepsilon^2_{3;12}, \}$. The generalized Gauss equations are:

$$\left\{\begin{array}{lcl} \mathrm{H}_{11}.\mathrm{H}_{22} - \mathrm{H}_{12}.\mathrm{H}_{21} & = & \mathcal{R}^1_{2;12} \\ \mathrm{H}_{11}.\mathrm{H}_{32} - \mathrm{H}_{12}.\mathrm{H}_{31} & = & \mathcal{R}^1_{3;12} \\ \mathrm{H}_{21}.\mathrm{H}_{32} - \mathrm{H}_{22}.\mathrm{H}_{31} & = & \mathcal{R}^2_{3:12} \end{array}\right. \tag{5.30}$$

When the Cartan identities are not taken into consideration, the differential of the generalized Gauss map $\mathcal{G}^3_{2,1}$ is:

$$\mathrm{d}\mathcal{G}^3_{2,1} = \begin{pmatrix} \mathrm{d}(\mathcal{G}^3_{2,1})^1_{2,12} \\ \mathrm{d}(\mathcal{G}^3_{2,1})^1_{3,12} \\ \mathrm{d}(\mathcal{G}^3_{2,1})^2_{3,12} \end{pmatrix} = \begin{pmatrix} \mathrm{H}_{22} & -\mathrm{H}_{12} & 0 & -\mathrm{H}_{21} & \mathrm{H}_{11} & 0 \\ \mathrm{H}_{32} & 0 & -\mathrm{H}_{12} & -\mathrm{H}_{31} & 0 & \mathrm{H}_{11} \\ 0 & \mathrm{H}_{32} & -\mathrm{H}_{22} & 0 & -\mathrm{H}_{31} & \mathrm{H}_{21} \end{pmatrix} . \begin{pmatrix} \mathrm{dH}_{11} \\ \mathrm{dH}_{21} \\ \mathrm{dH}_{31} \\ \mathrm{dH}_{12} \\ \mathrm{dH}_{22} \\ \mathrm{dH}_{32} \end{pmatrix} \tag{5.31}$$

If the generalized Cartan identities are taken into consideration, then

$$\mathrm{d}\mathcal{G}^3_{2,1} = \begin{pmatrix} \mathrm{H}_{22} - \psi^1_2\mathrm{H}_{21} & -\psi^i_2\mathrm{H}_{i1} - \psi^2_2\mathrm{H}_{21} & -\psi^3_2\mathrm{H}_{21} & \mathrm{H}_{11} & 0 \\ \mathrm{H}_{32} - \psi^1_2\mathrm{H}_{31} & -\psi^2_2\mathrm{H}_{31} & -\psi^i_2\mathrm{H}_{i1} - \psi^3_2\mathrm{H}_{31} & 0 & \mathrm{H}_{11} \\ 0 & \mathrm{H}_{32} & -\mathrm{H}_{22} & -\mathrm{H}_{31} & \mathrm{H}_{21} \end{pmatrix} . \begin{pmatrix} \mathrm{dH}_{11} \\ \mathrm{dH}_{21} \\ \mathrm{dH}_{31} \\ \mathrm{dH}_{22} \\ \mathrm{dH}_{32} \end{pmatrix} \tag{5.32}$$

Note that $\mathrm{H}_{i\lambda}$ are vectors in the Euclidean space $\mathcal{W}^3_{2,1}$ of dimension $\kappa^3_{2,1}$ which must be defined. We want to extract from the $\mathcal{W}^3_{2,1}$-valued matrix $\mathrm{d}\mathcal{G}^3_2$ a submatrix of maximal rank

(rank 3). Denote by L the subspace of the cotangent bundle of $\mathcal{W}^3_{2,1}\mathbb{R}^3 \otimes \mathbb{R}^2$ defined by $\mathrm{dH}_{11} = \mathrm{dH}_{21} = \mathrm{dH}_{31} = 0$. Then[3] $\mathrm{d}\mathcal{G}^3_{2,1}|_{\mathrm{L}}$ is :

$$\mathrm{d}\mathcal{G}^3_2|_{\mathrm{L}} = \begin{pmatrix} \mathrm{H}_{11} & 0 \\ 0 & \mathrm{H}_{11} \\ -\mathrm{H}_{31} & \mathrm{H}_{21} \end{pmatrix} \cdot \begin{pmatrix} \mathrm{dH}_{22} \\ \mathrm{dH}_{32} \end{pmatrix} \tag{5.33}$$

Therefore, if $\kappa^3_{2,1} \geqslant 2$, the matrix $\mathrm{d}\mathcal{G}^3_2|_{\mathrm{L}}$ is of maximal rank if H_{11} and H_{21} are linearly independent vectors of $\mathcal{W}^3_{2,1}$. For instance, if $\kappa^3_{2,1} = 2$, i.e., the normal direction are $a = 4, 5$, then

$$\mathrm{d}\mathcal{G}^3_2|_{\mathrm{L}} = \begin{pmatrix} \mathrm{H}^4_{11} & \mathrm{H}^5_{11} & 0 & 0 \\ 0 & 0 & \mathrm{H}^4_{11} & \mathrm{H}^5_{11} \\ -\mathrm{H}^4_{31} & -\mathrm{H}^5_{31} & \mathrm{H}^4_{21} & \mathrm{H}^5_{21} \end{pmatrix} \cdot \begin{pmatrix} \mathrm{dH}^4_{22} \\ \mathrm{dH}^5_{22} \\ \mathrm{dH}^4_{32} \\ \mathrm{dH}^5_{32} \end{pmatrix} \tag{5.34}$$

is of maximal rank if H_{11} and H_{21} are linearly independent vectors.

Before investigating the submersitivity of the generalized Gauss map, let us first define a flag of the subspaces of $\mathcal{K}^n_{m,m-1}$.

Flag of $\mathcal{K}^{\mathbf{n}}_{\mathbf{m,m-1}}$: Let us define the following subspaces of $\mathcal{K}^n_{m,m-1}$ as follows: for $k = 2, \dots, n$

$$\mathcal{E}^k|^n_{m,m-1} = \{\ (\mathcal{R}^i_{j;\lambda\mu}) \in \mathcal{K}^n_{m,m-1} | \mathcal{R}^i_{j;\lambda\mu} = 0, \text{if } 1 \leqslant i < j \leqslant k \text{ and } \forall 1 \leqslant \lambda < \mu \leqslant m\} \tag{5.35}$$

and for $\nu = 2, \dots, m$

$$\mathcal{E}_\nu|^n_{m,m-1} = \{\ (\mathcal{R}^i_{j;\lambda\mu}) \in \mathcal{K}^n_{m,m-1} | \mathcal{R}^i_{j;\lambda\mu} = 0, \text{if } 1 \leqslant \lambda < \mu \leqslant \nu \text{ and } \forall 1 \leqslant i < j \leqslant n\}. \tag{5.36}$$

By convention, $\mathcal{E}^1|^n_{m,m-1} = \mathcal{E}_1|^n_{m,m-1} = \mathcal{K}^n_{m,m-1}$. Therefore,

$$0 = \mathcal{E}^n|^n_{m,m-1} \subset \mathcal{E}^{n-1}|^n_{m,m-1} \subset \mathcal{E}^{n-2}|^n_{m,m-1} \subset \cdots \subset \mathcal{E}^2|^n_{m,m-1} \subset \mathcal{E}^1|^n_{m,m-1} = \mathcal{K}^n_{m,m-1} \tag{5.37}$$

$$0 = \mathcal{E}_m|^n_{m,m-1} \subset \mathcal{E}_{m-1}|^n_{m,m-1} \subset \mathcal{E}_{m-2}|^n_{m,m-1} \subset \cdots \subset \mathcal{E}_2|^n_{m,m-1} \subset \mathcal{E}_1|^1_{m,m-1} = \mathcal{K}^n_{m,m-1}. \tag{5.38}$$

Example 5.15 – $(\mathbb{V}^3, \mathcal{M}^4, \mathbf{g}, \nabla, \phi)_3$. An element in $\mathcal{K}^3_{4,3} = \wedge^2\mathbb{R}^3 \otimes \wedge^2\mathbb{R}^4 \simeq \mathbb{R}^{18}$ is:

$$\mathcal{R} = \begin{pmatrix} \mathcal{R}^1_{2;12} & \mathcal{R}^1_{2;13} & \mathcal{R}^1_{2;23} & \mathcal{R}^1_{2;14} & \mathcal{R}^1_{2;24} & \mathcal{R}^1_{2;34} \\ \mathcal{R}^1_{3;12} & \mathcal{R}^1_{3;13} & \mathcal{R}^1_{3;23} & \mathcal{R}^1_{3;14} & \mathcal{R}^1_{3;24} & \mathcal{R}^1_{3;34} \\ \mathcal{R}^2_{3;12} & \mathcal{R}^2_{3;13} & \mathcal{R}^2_{3;23} & \mathcal{R}^2_{3;14} & \mathcal{R}^2_{3;24} & \mathcal{R}^2_{3;34} \end{pmatrix} \tag{5.39}$$

and if $\mathcal{R}$ is in $\mathcal{E}^2|^3_{4,3}$ and in $\mathcal{E}^3|^3_{4,3}$ then respectively

$$\mathcal{R} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \mathcal{R}^1_{3;12} & \mathcal{R}^1_{3;13} & \mathcal{R}^1_{3;23} & \mathcal{R}^1_{3;14} & \mathcal{R}^1_{3;24} & \mathcal{R}^1_{3;34} \\ \mathcal{R}^2_{3;12} & \mathcal{R}^2_{3;13} & \mathcal{R}^2_{3;23} & \mathcal{R}^2_{3;14} & \mathcal{R}^2_{3;24} & \mathcal{R}^2_{3;34} \end{pmatrix} \text{ and } \mathcal{R} = (0) \tag{5.40}$$

and if $\mathcal{R}$ is in $\mathcal{E}_2|^3_{4,3}$, $\mathcal{E}_3|^3_{4,3}$ and in $\mathcal{E}_4|^3_{4,3}$ then respectively

$$\mathcal{R} = \begin{pmatrix} 0 & \mathcal{R}^1_{2;13} & \mathcal{R}^1_{2;23} & \mathcal{R}^1_{2;14} & \mathcal{R}^1_{2;24} & \mathcal{R}^1_{2;34} \\ 0 & \mathcal{R}^1_{3;13} & \mathcal{R}^1_{3;23} & \mathcal{R}^1_{3;14} & \mathcal{R}^1_{3;24} & \mathcal{R}^1_{3;34} \\ 0 & \mathcal{R}^2_{3;13} & \mathcal{R}^2_{3;23} & \mathcal{R}^2_{3;14} & \mathcal{R}^2_{3;24} & \mathcal{R}^2_{3;34} \end{pmatrix}, \mathcal{R} = \begin{pmatrix} 0 & 0 & 0 & \mathcal{R}^1_{2;14} & \mathcal{R}^1_{2;24} & \mathcal{R}^1_{2;34} \\ 0 & 0 & 0 & \mathcal{R}^1_{3;14} & \mathcal{R}^1_{3;24} & \mathcal{R}^1_{3;34} \\ 0 & 0 & 0 & \mathcal{R}^2_{3;14} & \mathcal{R}^2_{3;24} & \mathcal{R}^2_{3;34} \end{pmatrix},$$

and $\mathcal{R} = 0$.

[3] $\mathrm{d}\mathcal{G}^3_{2,1}|_{\mathrm{L}}$ is the submatrix of $\mathrm{d}\mathcal{G}^3_{2,1}$ defined by: $((\mathrm{d}\mathcal{G}^3_2(\partial/\partial\mathrm{H}_{22}))_a, (\mathrm{d}\mathcal{G}^3_2(\partial/\partial\mathrm{H}_{23}))_a)$.

THE CASE $(\mathbb{V}^{\mathbf{n}}, \mathcal{M}^{\mathbf{2}}, \mathbf{g}, \nabla, \phi)_{\mathbf{1}}$: [4] Recall that $\mathcal{K}^n_{2,1} = \wedge^2\mathbb{R}^n \otimes \mathbb{R}$. Some columns in the Jacobian of $\mathcal{G}^n_{2,1}$ are expressed as follows:

$$\text{for } k = 2, \ldots, n, \qquad \mathrm{d}\mathcal{G}^n_{2,1}\Big(\frac{\partial}{\partial \mathrm{H}^a_{k2}}\Big) = \Big(\sum_{i=1}^{k-1} \mathrm{H}^a_{i1}\varepsilon^i_{k;12} + (\text{terms in } \mathcal{E}^k|^n_{2,1})\Big) \in \mathcal{E}^{k-1}|^n_{2,1}. \tag{5.41}$$

Note that $\mathcal{E}^n|^n_{2,1} = 0$, and hence,

$$\mathrm{d}\mathcal{G}^n_{2,1}\Big(\frac{\partial}{\partial \mathrm{H}^a_{n2}}\Big) = \Big(\sum_{i=1}^{n-1} \mathrm{H}^a_{i1}\varepsilon^i_{n;12}\Big) \in \mathcal{E}^{n-1}|^n_{2,1}. \tag{5.42}$$

From the linear map $\mathrm{d}\mathcal{G}^n_{2,1}$, we want to extract a submatrix of maximal rank. Consider the submatrix $\Big((\mathrm{d}\mathcal{G}^n_{2,1}(\partial/\partial\mathrm{H}^a_{22}))_a, \ldots, (\mathrm{d}\mathcal{G}^n_{2,1}(\partial/\partial\mathrm{H}^a_{n2}))_a\Big)$. Each term $(\mathrm{d}\mathcal{G}^n_{2,1}(\partial/\partial\mathrm{H}^a_{k2}))_a$, for a fixed k, is a matrix with $n(n-1)/2$ lines and $\kappa^n_{2,1}$ columns. The equations (5.41), (5.42) and the inclusions (5.37) show that the submatrix $\Big((\mathrm{d}\mathcal{G}^n_{2,1}(\partial/\partial\mathrm{H}^a_{22}))_a, \ldots, (\mathrm{d}\mathcal{G}^n_{2,1}(\partial/\partial\mathrm{H}^a_{n2}))_a\Big)$ is of maximal rank if the vectors $\mathrm{H}_{11}, \mathrm{H}_{21}, \ldots \mathrm{H}_{(n-1)1}$ are linearly independent vectors of $\mathcal{W}^n_{2,1}$ and $\kappa^n_{2,1} \geqslant (n-1)$ where the minimal embedding codimension $\kappa^n_{2,1}$ is given by the dimension of $\mathcal{E}^{n-1}|^n_{2,1}$. Indeed, the matrix $\Big((\mathrm{d}\mathcal{G}^n_{2,1}(\partial/\partial\mathrm{H}^a_{22}))_a, \ldots, (\mathrm{d}\mathcal{G}^n_{2,1}(\partial/\partial\mathrm{H}^a_{n2}))_a\Big)$ is triangular by different sized blocks. This is due to the inclusions (5.37) of the spaces $\mathcal{E}^k|^n_{2,1}$. Note that the matrix $\Big((\mathrm{d}\mathcal{G}^n_{2,1}(\partial/\partial\mathrm{H}^a_{22}))_a, \ldots, (\mathrm{d}\mathcal{G}^n_{2,1}(\partial/\partial\mathrm{H}^a_{n2}))_a\Big)$ is rectangular, i.e., $n(n-1)/2$ lines and $(\kappa^n_{2,1} \times (n-1))$ columns. There are actually $(n-1)$ terms in the "diagonal" and they all have the same number of columns $\kappa^n_{2,1}$. The first term of the "diagonal" has one line and obviously starts at the first line, the second term has 2 lines and is at the second line, the third term has 3 lines and starts at the line number $1+2 = 3$, …, and the last term has $(n-1)$ lines and starts at the line number $(n-2)(n-1)/2$. From (5.41) and (5.42), the "diagonal" of $\Big((\mathrm{d}\mathcal{G}^n_{2,1}(\partial/\partial\mathrm{H}^a_{22}))_a, \ldots, (\mathrm{d}\mathcal{G}^n_{2,1}(\partial/\partial\mathrm{H}^a_{n2}))_a\Big)$ is: $\mathrm{diag}\Big((\mathrm{H}^a_{11})_a, {}^t(\mathrm{H}^a_{11}, \mathrm{H}^a_{21})_a, \ldots, {}^t(\mathrm{H}^a_{11}, \ldots, \mathrm{H}^a_{(n-1)1})_a\Big)$, and since $0 \subset \mathcal{E}^{n-1} \subset \mathcal{E}^{n-2} \subset \cdots \subset \mathcal{E}^2 \subset \mathcal{E}^1 = \mathcal{K}^n_{2,1}$, the terms above this "diagonal" vanish in the matrix $\Big((\mathrm{d}\mathcal{G}^n_{2,1}(\partial/\partial\mathrm{H}^a_{22}))_a, \ldots, (\mathrm{d}\mathcal{G}^n_{2,1}(\partial/\partial\mathrm{H}^a_{n2}))_a\Big)$. Note that ${}^t(\mathrm{H}_{11}, \ldots, \mathrm{H}_{k1})_a$ is a matrix with k lines and $\kappa^n_{2,1}$ columns. The condition of being linearly independent for the vector $(\mathrm{H}_{11}, \ldots \mathrm{H}_{(n-1)1})$ assures that one can always extract, for each term of the diagonal, a submatrix of maximal rank. For instance, the "diagonal" term of $\mathrm{d}\mathcal{G}^n_{2,1}(\partial/\partial\mathrm{H}^a_{32})$ is ${}^t(\mathrm{H}^a_{11}, \mathrm{H}^a_{21})$, which is a $2 \times \kappa^n_{2,1}$ matrix, and since the two vectors are linearly independent, there exists an invertible 2×2 submatrix. The same argument holds for each term of the "diagonal", and finally, $\kappa^n_{2,1} \geqslant \dim(\mathcal{E}^{n-1}|^n_{2,1})$ assures that the last terms of the "diagonal", $(\mathrm{d}\mathcal{G}^n_{2,1}(\partial/\partial\mathrm{H}^a_{n2}))_a$, are of maximal rank.

Another explanation and interpretation. Only in this subparagraph, we will change some notations in order to give the reader another way of interpreting the different equations and objects. It can also be applied to the next case, i.e., the case $(\mathbb{V}^2, \mathcal{M}^m, g, \nabla, \phi)_{m-1})$.

[4] After proving the submersitivity of the generalized Gauss map in this case, we provide another explanation and interpretation of the spaces and equations using other notations and conventions.

The curvature tensor $\mathcal{R}$ in $\mathcal{K}^n_{2,1}$ is an element of $\mathfrak{so}(n) \otimes \wedge^2\mathbb{R}^2 \simeq \mathfrak{so}(n)$. Hence , $\mathcal{R}$ can be expressed as follows:

$$\begin{pmatrix} 0 & \mathcal{R}^1_{2;12} & \mathcal{R}^1_{3;12} & \mathcal{R}^1_{4;12} & \cdots & \mathcal{R}^1_{n;12} \\ -\mathcal{R}^1_{2;12} & 0 & \mathcal{R}^2_{3;12} & \mathcal{R}^2_{4;12} & & \vdots \\ -\mathcal{R}^1_{3;12} & -\mathcal{R}^2_{3;12} & 0 & \mathcal{R}^3_{4;12} & \ddots & \\ -\mathcal{R}^1_{4;12} & -\mathcal{R}^2_{4;12} & -\mathcal{R}^3_{4;12} & 0 & \ddots & \mathcal{R}^{n-1}_{n;12} \\ \vdots & & & \ddots & \ddots & \\ -\mathcal{R}^1_{n;12} & \cdots & & & \mathcal{R}^{n-1}_{n;12} & 0 \end{pmatrix} \tag{5.43}$$

Notice that if $\mathcal{R}$ is in the subspace $\mathcal{E}^k|^n_{2,1}$ defined previously, then the matrix (5.43) has an $k \times k$ vanishing sub-matrix located at the top left edge of the matrix (5.43). Indeed, the elements R in $\mathcal{E}^2|^n_{2,1}$ and in $\mathcal{E}^3|^n_{2,1}$ are respectively

$$\begin{pmatrix} 0 & 0 & \mathcal{R}^1_{3;12} & \mathcal{R}^1_{4;12} & \cdots & \mathcal{R}^1_{n;12} \\ 0 & 0 & \mathcal{R}^2_{3;12} & \mathcal{R}^2_{4;12} & & \vdots \\ -\mathcal{R}^1_{3;12} & -\mathcal{R}^2_{3;12} & 0 & \mathcal{R}^3_{4;12} & \ddots & \\ -\mathcal{R}^1_{4;12} & -\mathcal{R}^2_{4;12} & -\mathcal{R}^3_{4;12} & 0 & \ddots & \mathcal{R}^{n-1}_{n;12} \\ \vdots & & & \ddots & \ddots & \\ -\mathcal{R}^1_{n;12} & \cdots & & & \mathcal{R}^{n-1}_{n;12} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0^3 & \mathcal{R}^1_{4;12} & \cdots & \mathcal{R}^1_{n;12} \\ 0 & 0 & 0 & \mathcal{R}^2_{4;12} & & \vdots \\ 0 & 0 & 0 & \mathcal{R}^3_{4;12} & \ddots & \\ -\mathcal{R}^1_{4;12} & -\mathcal{R}^2_{4;12} & -\mathcal{R}^3_{4;12} & 0 & \ddots & \mathcal{R}^{n-1}_{n;12} \\ \vdots & & & \ddots & \ddots & \\ -\mathcal{R}^1_{n;12} & \cdots & & & \mathcal{R}^{n-1}_{n;12} & 0 \end{pmatrix}.$$

Recall that the generalized Gauss map $\mathcal{G}^n_{2,1}$ is defined by $\mathcal{G}^n_{2,1}(\mathrm{H}) = (\mathrm{H}_{i1}\mathrm{H}_{j2} - \mathrm{H}_{i2}\mathrm{H}_{j1})^i_j$ which is an element of $\mathfrak{so}(n)$. The differential of the generalized Gauss map is then

$$\mathrm{d}\mathcal{G}^n_{2,1} = (\mathrm{H}_{i1}\mathrm{d}\mathrm{H}_{j2} + \mathrm{H}_{j2}\mathrm{d}\mathrm{H}_{i1} - \mathrm{H}_{i2}\mathrm{d}\mathrm{H}_{j1} - \mathrm{H}_{j1}\mathrm{d}\mathrm{H}_{i2})^i_j. \tag{5.44}$$

Then, for $k = 2, \ldots, n$, the relation (5.41) becomes

$$\mathrm{d}\mathcal{G}^n_{2,1}\Big(\frac{\partial}{\partial \mathrm{H}^a_{k2}}\Big) = \begin{pmatrix} 0 & \cdots & 0 & \mathrm{H}^a_{11} & * & \cdots & * \\ \vdots & & \vdots & \mathrm{H}^a_{21} & \vdots & & \vdots \\ & & & \vdots & & & \\ 0 & \cdots & 0 & \mathrm{H}^a_{(k-1)1} & & & \\ -\mathrm{H}^a_{11} & -\mathrm{H}^a_{21} \cdots & -\mathrm{H}^a_{(k-1)1} & 0 & * & & \\ * & \cdots & & * & \ddots & \ddots & \\ \vdots & & & & \ddots & \ddots & * \\ * & \cdots & & & & * & 0 \end{pmatrix} \in \mathcal{E}^k|^n_{2,1}. \tag{5.45}$$

In particular, (5.42) has the following form:

$$\mathrm{d}\mathcal{G}^n_{2,1}\Big(\frac{\partial}{\partial \mathrm{H}^a_{n2}}\Big) = \begin{pmatrix} 0 & \cdots & & 0 & \mathrm{H}^a_{11} \\ \vdots & & & \vdots & \mathrm{H}^a_{21} \\ & & & & \vdots \\ & & & & \mathrm{H}^a_{(n-2)1} \\ 0 & \cdots & & 0 & \mathrm{H}^a_{(n-1)1} \\ -\mathrm{H}^a_{11} & -\mathrm{H}^a_{21} \cdots & -\mathrm{H}_{(n-2)1} & -\mathrm{H}^a_{(n-1)1} & 0 \end{pmatrix} \in \mathcal{E}^{n-1}|^n_{2,1}. \tag{5.46}$$

THE CASE $(\mathbb{V}^{\mathbf{2}},\mathcal{M}^{\mathbf{m}},\mathbf{g},\nabla,\phi)_{\mathbf{m-1}}$: Recall that $\mathcal{K}^2_{m,m-1}=\mathbb{R}\otimes\wedge^2\mathbb{R}^n$. Some columns in the Jacobian of $\mathcal{G}^2_{m,m-1}$ are expressed as follows:

$$\text{for } \nu=2,\dots,m,\qquad \mathrm{d}\mathcal{G}^2_{m,m-1}\Big(\frac{\partial}{\partial \mathrm{H}^a_{2\nu}}\Big)=\Big(\sum_{\lambda=1}^{\nu-1}\mathrm{H}^a_{1\lambda}\varepsilon^1_{2;\lambda\nu}+(\text{ terms in }\mathcal{E}_\nu|^2_{m,m-1})\Big)\in\mathcal{E}_{\nu-1}|^2_{m,m-1}. \tag{5.47}$$

Similarly, note that $\mathcal{E}_m|^2_{m,m-1}=0$ and hence

$$\mathrm{d}\mathcal{G}^2_{m,m-1}\Big(\frac{\partial}{\partial \mathrm{H}^a_{2m}}\Big)=\Big(\sum_{\lambda=1}^{m-1}\mathrm{H}^a_{1\lambda}\varepsilon^1_{2;\lambda m}\Big)\in\mathcal{E}_{m-1}|^2_{m,m-1}. \tag{5.48}$$

Similarly, from the linear map $\mathrm{d}\mathcal{G}^2_{m,m-1}$, we want to extract a submatrix of maximal rank. Consider the submatrix $\Big((\mathrm{d}\mathcal{G}^2_{m,m-1}(\partial/\partial\mathrm{H}^a_{22}))_a,\dots,(\mathrm{d}\mathcal{G}^2_{m,m-1}(\partial/\partial\mathrm{H}^a_{n2m}))_a\Big)$. Each term $(\mathrm{d}\mathcal{G}^2_{m,m-1}(\partial/\partial\mathrm{H}^a_{2\nu}))_a$, for a fixed ν, is a matrix with $m(m-1)/2$ lines and $\kappa^2_{m,m-1}$ columns. The equations (5.47), (5.48) and the inclusions (5.38) show that the submatrix $\Big((\mathrm{d}\mathcal{G}^2_{m,m-1}(\partial/\partial\mathrm{H}^a_{22}))_a,\dots,(\mathrm{d}\mathcal{G}^2_{m,m-1}(\partial/\partial\mathrm{H}^a_{2m}))_a\Big)$ is of maximal rank if the vectors $\mathrm{H}_{11},\mathrm{H}_{12},\dots\mathrm{H}_{1(m-1)}$ are linearly independent vectors of $\mathcal{W}^2_{m,m-1}$ and $\kappa^2_{m,m-1}\geqslant(m-1)$ where the minimal embedding codimension $\kappa^2_{m,m-1}$ is given by the dimension of $\mathcal{E}_{m-1}|^2_{m,m-1}$. Indeed, the matrix $\Big((\mathrm{d}\mathcal{G}^2_{m,m-1}(\partial/\partial\mathrm{H}^a_{22}))_a,\dots,(\mathrm{d}\mathcal{G}^2_{m,m-1}(\partial/\partial\mathrm{H}^a_{2m}))_a\Big)$ is triangular by different sized blocks. This is due to the inclusions (5.38) of the spaces $\mathcal{E}_\nu|^2_{m,m-1}$. Note that $\Big((\mathrm{d}\mathcal{G}^2_{m,m-1}(\partial/\partial\mathrm{H}^a_{22}))_a,\dots,(\mathrm{d}\mathcal{G}^2_{m,m-1}(\partial/\partial\mathrm{H}^a_{2m}))_a\Big)$ is rectangular, i.e., $n(n-1)/2$ lines and $(\kappa^2_{m,m-1}\times(m-1))$ columns. There are actually $(m-1)$ terms in the "diagonal" and they all have the same number of columns $\kappa^2_{m,m-1}$. The first term of the "diagonal" has one line and obviously starts at the first line, the second term has 2 lines and is at the second line, the third term has 3 lines and starts at the line number $1+2=3$, …, and the last term has $(m-1)$ lines and starts at the line number $(m-2)(m-1)/2$. From (5.47) and (5.48), the "diagonal" of $\Big((\mathrm{d}\mathcal{G}^2_{m,m-1}(\partial/\partial\mathrm{H}^a_{22}))_a,\dots,(\mathrm{d}\mathcal{G}^2_{m,m-1}(\partial/\partial\mathrm{H}^a_{2m}))_a\Big)$ is: $\mathrm{diag}\Big((\mathrm{H}^a_{11})_a,{}^t(\mathrm{H}^a_{11},\mathrm{H}^a_{12})_a,\dots,{}^t(\mathrm{H}^a_{11},\dots,\mathrm{H}^a_{1(m-1)})_a\Big)$, and since $0\subset\mathcal{E}_{m-1}\subset\mathcal{E}_{n-2}\subset\cdots\subset\mathcal{E}_2\subset\mathcal{E}_1=\mathcal{K}^n_{2,1}$, the terms above this "diagonal" vanish in the matrix $\Big((\mathrm{d}\mathcal{G}^2_{m,m-1}(\partial/\partial\mathrm{H}^a_{22}))_a,\dots,(\mathrm{d}\mathcal{G}^2_{m,m-1}(\partial/\partial\mathrm{H}^a_{2m}))_a\Big)$. Note that ${}^t(\mathrm{H}_{11},\dots,\mathrm{H}_{1\nu})_a$ is a matrix with ν lines and $\kappa^2_{m,m-1}$ columns. The condition of being linearly independent for the vector $(\mathrm{H}_{11},\dots\mathrm{H}_{1(m-1)})$ assures that one can always extract, for each term of the diagonal, a submatrix of maximal rank. For instance, the "diagonal" term of $\mathrm{d}\mathcal{G}^2_{m,m-1}(\partial/\partial\mathrm{H}^a_{24})$, is ${}^t(\mathrm{H}^a_{11},\mathrm{H}^a_{12},\mathrm{H}^a_{13})$, which is a $3\times\kappa^2_{m,m-1}$ matrix, and since the three vectors are linearly independent, there exists an invertible 3×3 submatrix. The same argument holds for each term of the "diagonal", and finally, $\kappa^2_{m,m-1}\geqslant\dim(\mathcal{E}_{m-1}|^2_{m,m-1})$ assures that the last terms of the "diagonal", $(\mathrm{d}\mathcal{G}^2_{m,m-1}(\partial/\partial\mathrm{H}^a_{2m}))_a$, are of maximal rank.

THE CASE $(\mathbb{V}^{\mathbf{n}},\mathcal{M}^{\mathbf{m}},\mathbf{g},\nabla,\phi)_{\mathbf{m-1}}$: For the general conservation laws case, we define the following subspaces of $\mathcal{K}^n_{m,m-1}$: for $k=2,\dots,n$ and for $\nu=2,\dots,m$,

$$\mathcal{E}^k_\nu|^n_{m,m-1}=\{(\mathcal{R})^i_{j;\lambda\mu}\in\mathcal{K}^n_{m,m-1}|\mathcal{R}^i_{j;\lambda\mu}=0,\ \text{if } 1\leqslant i<j\leqslant k \text{ and } 1\leqslant\lambda<\mu\leqslant\nu\} \tag{5.49}$$

and hence,

$$\mathcal{E}_\nu^n|^n_{m,m-1} = \mathcal{E}_\nu|^n_{m,m-1} \quad \text{and} \quad \mathcal{E}^k_m|^n_{m,m-1} = \mathcal{E}^k|^n_{m,m-1}. \tag{5.50}$$

By convention, $\mathcal{E}^1_\nu|^n_{m,m-1} = \mathcal{K}^n_{m,m-1}$ and $\mathcal{E}^k_1|^n_{m,m-1} = \mathcal{K}|^n_{m,m-1}$.

Remark 5.16 Let us ν and k be fixed. We have the same kind of flags as in (5.37) and (5.38)

$$\mathcal{E}_\nu|^n_{m,m-1} = \mathcal{E}^n_\nu|^n_{m,m-1} \subset \mathcal{E}^{n-1}_\nu|^n_{m,m-1} \subset \mathcal{E}^{n-2}_\nu|^n_{m,m-1} \subset \cdots \subset \mathcal{E}^2_\nu|^n_{m,m-1} \subset \mathcal{E}^1_\nu|^n_{m,m-1} = \mathcal{K}^n_{m,m-1}$$
$$\mathcal{E}^k|^n_{m,m-1} = \mathcal{E}^k_m|^n_{m,m-1} \subset \mathcal{E}^k_{m-1}|^n_{m,m-1} \subset \mathcal{E}^k_{m-2}|^n_{m,m-1} \subset \cdots \subset \mathcal{E}^k_2|^n_{m,m-1} \subset \mathcal{E}^k_1|^1_{m,m-1} = \mathcal{K}^n_{m,m-1}.$$

Example 5.17 – $(\mathbb{V}^3, \mathcal{M}^4, \mathbf{g}, \nabla, \phi)_3$-Continued. $\mathcal{E}^2_4|^3_{4,3} = \mathcal{E}^2|^3_{4,3}$, $\mathcal{E}^3_2|^3_{4,3} = \mathcal{E}_2|^3_{4,3}$,$\mathcal{E}^3_3|^3_{4,3} = \mathcal{E}_3|^3_{4,3}$ and $\mathcal{E}^3_4|^3_{4,3} = 0$ and if $\mathcal{R}$ is in $\mathcal{E}^2_2|^3_{4,3}$, $\mathcal{E}^2_3|^3_{4,3}$, then respectively

$$\mathcal{R} = \begin{pmatrix} 0 & \mathcal{R}^1_{2;13} & \mathcal{R}^1_{2;23} & \mathcal{R}^1_{2;14} & \mathcal{R}^1_{2;24} & \mathcal{R}^1_{2;34} \\ \mathcal{R}^1_{3;12} & \mathcal{R}^1_{3;13} & \mathcal{R}^1_{3;23} & \mathcal{R}^1_{3;14} & \mathcal{R}^1_{3;24} & \mathcal{R}^1_{3;34} \\ \mathcal{R}^2_{3;12} & \mathcal{R}^2_{3;13} & \mathcal{R}^2_{3;23} & \mathcal{R}^2_{3;14} & \mathcal{R}^2_{3;24} & \mathcal{R}^2_{3;34} \end{pmatrix}, \tag{5.51}$$

$$\mathcal{R} = \begin{pmatrix} 0 & 0 & 0 & \mathcal{R}^1_{2;14} & \mathcal{R}^1_{2;24} & \mathcal{R}^1_{2;34} \\ \mathcal{R}^1_{3;12} & \mathcal{R}^1_{3;13} & \mathcal{R}^1_{3;23} & \mathcal{R}^1_{3;14} & \mathcal{R}^1_{3;24} & \mathcal{R}^1_{3;34} \\ \mathcal{R}^2_{3;12} & \mathcal{R}^2_{3;13} & \mathcal{R}^2_{3;23} & \mathcal{R}^2_{3;14} & \mathcal{R}^2_{3;24} & \mathcal{R}^2_{3;34} \end{pmatrix}. \tag{5.52}$$

Proposition 5.18 – Extension of (5.38) For $(\mathbb{V}^n, \mathcal{M}^m, g, \nabla, \phi)_{m-1}$, we can have a longer flag by replacing in (5.38) each inclusion of the type $\mathcal{E}_\nu|^n_{m,m-1} \subset \mathcal{E}_{(\nu-1)}|^n_{m,m-1}$, for $\nu = 2, \ldots, m$, by

$$\mathcal{E}_\nu \subset \left(\mathcal{E}_{(\nu-1)} \cap \mathcal{E}^{n-1}_\nu\right) \subset \left(\mathcal{E}_{(\nu-1)} \cap \mathcal{E}^{n-2}_\nu\right) \subset \cdots \subset \left(\mathcal{E}_{(\nu-1)} \cap \mathcal{E}^3_\nu\right) \subset \left(\mathcal{E}_{(\nu-1)} \cap \mathcal{E}^2_\nu\right) \subset \mathcal{E}_{(\nu-\mathbf{1})} \tag{5.53}$$

Note that we dropped $|^n_{m,m-1}$ for each subspace $\mathcal{E}$, in the above equation, for more clarity.

Example 5.19 – $(\mathbb{V}^4, \mathcal{M}^5, \mathbf{g}, \nabla, \phi)_4$. We drop in this example the signs $|^4_{5,4}$ next to the subspaces $\mathcal{E}^k_\nu|^4_{5,4}$. When we put (5.53) in (5.38), we obtain $0 = \mathcal{E}_5 \subset \left(\mathcal{E}_4 \cap \mathcal{E}^3_5\right) \subset \left(\mathcal{E}_4 \cap \mathcal{E}^2_5\right) \subset \mathcal{E}_4 \subset \left(\mathcal{E}_3 \cap \mathcal{E}^3_4\right) \subset \left(\mathcal{E}_3 \cap \mathcal{E}^2_4\right) \subset \mathcal{E}_3 \subset \left(\mathcal{E}_2 \cap \mathcal{E}^3_3\right) \subset \left(\mathcal{E}_2 \cap \mathcal{E}^2_3\right) \subset \mathcal{E}_2 \subset \mathcal{E}^3_2 \subset \mathcal{E}^2_2 \subset \mathcal{E}_1 = \mathcal{K}^4_{5,4}$.

Using Proposition 5.53, the inclusion flag for the generalized curvature space in the conservation law case is the following (the signs $|^n_{m,m-1}$ next to the space $\mathcal{E}$ are dropped for more clarity):

$$\begin{aligned} 0 &\subset (\mathcal{E}_{m-1} \cap \mathcal{E}^{n-1}) \subset (\mathcal{E}_{m-1} \cap \mathcal{E}^{n-2}) \subset \cdots \subset (\mathcal{E}_{m-1} \cap \mathcal{E}^3) \subset (\mathcal{E}_{m-1} \cap \mathcal{E}^2) \subset \\ \mathcal{E}_{\mathbf{m-1}} &\subset (\mathcal{E}_{m-2} \cap \mathcal{E}^{n-1}_{m-1}) \subset (\mathcal{E}_{m-2} \cap \mathcal{E}^{n-2}_{m-1}) \subset \cdots \subset (\mathcal{E}_{m-2} \cap \mathcal{E}^3_{m-1}) \subset (\mathcal{E}_{m-2} \cap \mathcal{E}^2_{m-1}) \subset \\ \mathcal{E}_{\mathbf{m-2}} &\subset (\mathcal{E}_{m-3} \cap \mathcal{E}^{n-1}_{m-2}) \subset (\mathcal{E}_{m-3} \cap \mathcal{E}^{n-2}_{m-2}) \subset \cdots \subset (\mathcal{E}_{m-3} \cap \mathcal{E}^3_{m-2}) \subset (\mathcal{E}_{m-3} \cap \mathcal{E}^2_{m-2}) \subset \\ &\vdots \\ \mathcal{E}_{\mathbf{3}} &\subset (\mathcal{E}_2 \cap \mathcal{E}^{n-1}_3) \subset (\mathcal{E}_2 \cap \mathcal{E}^{n-2}_3) \subset \cdots \subset (\mathcal{E}_2 \cap \mathcal{E}^3_3) \subset (\mathcal{E}_2 \cap \mathcal{E}^2_3) \subset \\ \mathcal{E}_{\mathbf{2}} &\subset \mathcal{E}^{n-1}_2 \subset \mathcal{E}^{n-2}_2 \subset \cdots \subset \mathcal{E}^3_2 \subset \mathcal{E}^2_2 \subset \mathcal{K}^n_{m,m-1}. \end{aligned}$$

We proceed in the same way to prove lemma 5.14. The inclusion of the spaces $\mathcal{E}^k_\nu|^n_{m,m-1}$ is more complex and is given by the Proposition 5.18. We have, for $k = 2, \ldots, n$ and $\nu = 2, \ldots, m$

$$\mathrm{d}\mathcal{G}^n_{m,m-1}(\partial/\partial \mathrm{H}^a_{k\nu}) = \Big(\sum_{\substack{i = 1, \ldots, k-1 \\ \lambda = 1, \ldots \nu - 1}} \mathrm{H}^a_{i\lambda} \varepsilon^i_{k;\lambda\nu} + (\text{terms in } \mathcal{E}^{k+1}_{\nu-1})\Big) \in \mathcal{E}^{k-1}_{\nu-1}|^n_{m,m-1} \tag{5.54}$$

and since $\mathcal{E}_k^n|_{m,m-1}^n = 0$,

$$\mathrm{d}\mathcal{G}_{m,m-1}^n(\partial/\partial \mathrm{H}_{nm}^a) = \Big(\sum_{\substack{i=1,\dots,n-1\\ \lambda=1,\dots m-1}} \mathrm{H}_{i\lambda}^a \varepsilon_{n;\lambda m}^i\Big) \in \mathcal{E}_{m-1}^{n-1}|_{m,m-1}^n. \tag{5.55}$$

As we explained previously, from the linear map $\mathrm{d}\mathcal{G}_{m,m-1}^n$, we want to extract a submatrix of maximal rank. Consider the submatrix

$$\Big((\mathrm{d}\mathcal{G}_{m,m-1}^n(\partial/\partial \mathrm{H}_{22}^a))_a, \dots (\mathrm{d}\mathcal{G}_{m,m-1}^n(\partial/\partial \mathrm{H}_{n2}^a))_a, \dots, (\mathrm{d}\mathcal{G}_{m,m-1}^n(\partial/\partial \mathrm{H}_{2m}^a))_a, \dots, (\mathrm{d}\mathcal{G}_{m,m-1}^n(\partial/\partial \mathrm{H}_{nm}^a))_a\Big)$$

which has $n(n-1)m(m-1)/4$ lines and $\kappa_{m,m-1}^n \times (n-1)(m-1)$ columns. This matrix is of maximal rank if the vectors $(\mathrm{H}_{i\lambda})_{i=1,\dots,(n-1) \text{ and } \lambda=1,\dots,m-1}$ are linearly independent vectors of $\mathcal{W}_{m,m-1}^n$ where $\kappa_{m,m-1}^n \geqslant (n-1)(m-1)$. The minimal embedding codimension is given by the dimension of $(\mathcal{E}^{n-1} \cap \mathcal{E}_{m-1}|_{m,m-1}^n)$. Indeed, Proposition (5.18) shows that the submatrix is triangular by different sized blocks and that the terms above the block-diagonal are zero. There are $(n-1)(m-1)$ terms in the "diagonal" and they have the same number of columns κ_{m-1}^n.

The surjectivity of the generalized Gauss map

It remains to show that the generalized Gauss map is surjective, namely

$$\mathcal{G}_{m,m-1}^n(\mathcal{H}_{m,m-1}^n) = \mathcal{K}_{m,m-1}^n. \tag{5.56}$$

It is sufficient to show that there exists a pre-image of 0, i.e., vectors $\mathrm{H}_{i\lambda}$ in $\mathcal{W}_{m,m-1}^n$, satisfying generalized Cartan identities and such that the set $\{\mathrm{H}_{i\lambda}\}$ for $i = 1, \dots, n-1$ and $\lambda = 1, \dots, m-1$ are linearly independent vectors in $\mathcal{W}_{m,m-1}^n$. Indeed, the differential of the generalized Gauss map being surjective implies that $\mathcal{G}_{m,m-1}^n(\mathcal{H}_{m,m-1}^n)$ will contain a neighborhood of 0 in $\mathcal{K}_{m,m-1}^n$, and thus $\mathcal{G}_{m,m-1}^n(\mathcal{H}_{m,m-1}^n) = \mathcal{K}_{m,m-1}^n$ as $\mathcal{G}_{m,m-1}^n(\rho \mathrm{H}) = \rho^2 \mathcal{G}_{m,m-1}^n(\mathrm{H})$.

We will construct a pre-image of 0 in $\mathcal{H}_{m,m-1}^n$. Recall that $\mathcal{W}_{m,m-1}^n$ is of dimension $\kappa_{m,m-1}^n \geqslant (n-1)(m-1)$. We can choose $\mathrm{H}_{i\lambda}$ as follows:

$$\{\mathrm{H}_{i\lambda}\}_{i=1,\dots,n-1 \text{ and } \lambda=1,\dots,m-1} \text{ is an orthonormal set of vectors in } \mathcal{W}_{m,m-1}^n \tag{5.57}$$

$$\mathrm{H}_{n1} = \mathrm{H}_{n2} = \cdots = \mathrm{H}_{nm} = 0 \tag{5.58}$$

$$\text{For } j = 2, \dots, m, \qquad \mathrm{H}_{jm} = \sum_{\substack{i=1,\dots,n-1\\ \lambda=1,\dots,m-1}} \mathrm{A}_j^{i\lambda} \mathrm{H}_{i\lambda} \tag{5.59}$$

where

$$\mathrm{A}_j^{1\lambda} = \psi_{\Lambda \smallsetminus \lambda}^j \qquad \text{and} \qquad \mathrm{A}_j^{i\lambda} = \mathrm{A}_i^{j\lambda}. \tag{5.60}$$

The vectors in (5.59) are expressed as follows

$$\begin{pmatrix} \mathrm{H}_{2m} \\ \vdots \\ \mathrm{H}_{(n-1)m} \end{pmatrix} = \begin{pmatrix} \psi^2_{\Lambda\smallsetminus 1} & \mathrm{A}^{21}_2 \cdots \mathrm{A}^{(n-1)1}_2 & \cdots & \psi^2_{\Lambda\smallsetminus(m-1)} & \mathrm{A}^{2(m-1)}_2 \cdots \mathrm{A}^{(n-1)(m-1)}_2 \\ \vdots & \vdots \quad \vdots & & & \vdots \quad \vdots \\ \psi^{n-1}_{\Lambda\smallsetminus 1} & \mathrm{A}^{21}_{(n-1)} \cdots \mathrm{A}^{(n-1)1}_{(n-1)} & \cdots & \psi^{n-1}_{\Lambda\smallsetminus(m-1)} & \mathrm{A}^{2(m-1)}_{(n-1)} \cdots \mathrm{A}^{(n-1)(m-1)}_{(n-1)} \end{pmatrix} \begin{pmatrix} \mathrm{H}_{11} \\ \mathrm{H}_{21} \\ \vdots \\ \mathrm{H}_{(n-1)1} \\ \mathrm{H}_{12} \\ \mathrm{H}_{22} \\ \vdots \\ \mathrm{H}_{(n-1)2} \\ \vdots \\ \mathrm{H}_{1(m-1)} \\ \mathrm{H}_{2(m-1)} \\ \vdots \\ \mathrm{H}_{(n-1)(m-1)} \end{pmatrix}$$

5.2.2 ANOTHER PROOF OF THEOREM 4.12

This proof is based on constructing explicitly an ordinary m-integral element, and Cartan characters are computed by expliciting the polar space of an integral flag. As defined above, let us consider $\mathcal{I}^n_{m,m-1}$ to be an exterior ideal on $\mathbf{\Sigma}^n_{m,m-1}$. Let us denote by (X_λ) the dual basis of (η^λ) and by (Y_A) the dual basis of $(\varpi^\mathrm{A}) = (\varpi^{\sigma\binom{i}{j}}, \varpi^{\sigma\binom{a}{j}}) = (\omega^i_j - \eta^i_j, \omega^a_i)$ where $\mathrm{A} = 1, \ldots, \dim \mathbf{\Sigma}^n_{m,m-1} - m$ and $\sigma\binom{i}{j} = (j-i) + \dfrac{n(n-1)}{2} - \dfrac{(n-i)(n-i+1)}{2}$ for $1 \leqslant i < j \leqslant n$ and $\sigma\binom{a}{i} = \dfrac{n(n-1)}{2} + (a-n-1)n + i$ for $i = 1, \ldots, n$ and $a = n+1, \ldots, n + \kappa^n_{m,m-1}$. Let us consider on the Grassmannian manifold $\mathrm{G}_m(\mathbf{\Sigma}^n_{m,m-1}, \eta^\Lambda)$ a basis $\mathfrak{X}_\lambda$ defined as follows:

$$\mathfrak{X}_\lambda(\mathrm{E}) = \mathrm{X}_\lambda + \mathrm{P}^\mathrm{A}_\lambda(\mathrm{E})\mathrm{Y}_\mathrm{A}, \qquad \mathrm{A} = 1, \ldots, \dim \mathbf{\Sigma}^n_{m,m-1} - m. \tag{5.61}$$

Let $(\Pi^\lambda(\mathrm{E}))$ be the dual basis of $(\mathfrak{X}_\lambda(\mathrm{E}))$. In order to compute the codimension in the Grassmannian $\mathrm{G}_m(\mathrm{T}\mathbf{\Sigma}^n_{m,m-1}, \eta^\Lambda)$ of m-integral elements of $\mathcal{I}^n_{m,m-1}$, we pull back the forms that generate the exterior ideal. To do so, we evaluate the forms on the basis $\mathfrak{X}_\lambda(\mathrm{E})$ and hence the expression of the forms on the Grassmannian are:

$$(\varpi^{\sigma\binom{i}{j}})_\mathrm{E} = \mathrm{P}^{\sigma\binom{i}{j}}_\lambda \Pi^\lambda \tag{5.62}$$

$$\Big(\sum_a \varpi^{\sigma\binom{a}{i}} \wedge \varpi^{\sigma\binom{a}{j}} - \Omega^i_j\Big)_\mathrm{E} = \Big(\sum_a \mathrm{P}^{\sigma|^a_i}_\lambda \mathrm{P}^{\sigma\binom{a}{j}}_\mu - \mathrm{P}^{\sigma\binom{a}{i}}_\mu \mathrm{P}^{\sigma\binom{a}{j}}_\lambda - \mathcal{R}^i_{j;\lambda\mu}\Big)\Pi^\lambda \wedge \Pi^\mu \tag{5.63}$$

$$(\varpi^{\sigma\binom{a}{j}} \wedge \phi^i)_\mathrm{E} = \Big(\sum_\lambda (-1)^{\lambda+1} \psi^i_{\Lambda\smallsetminus\lambda} \mathrm{P}^{\sigma\binom{a}{i}}_\lambda\Big)\Pi^\Lambda. \tag{5.64}$$

The number of functions that have linearly independent differentials represents the desired codimension and hence with lemma 5.14

$$\operatorname{codim} \mathcal{V}_m(\mathcal{I}^n_{m,m-1}, \eta^\Lambda) = m\frac{n(n-1)}{2} + \frac{n(n-1)}{2}\frac{m(m-1)}{2} + \kappa^n_{m,m-1} \tag{5.65}$$

We will now construct an explicit ordinary m-integral element of $\mathcal{I}^n_{m,m-1}$. A tangent vector ξ in the tangent space of $\mathbf{\Sigma}^n_{m,m-1}$ is expressed as follows:

$$\xi = \xi^\lambda_\mathcal{M}\mathrm{X}_\lambda + \xi^\mathrm{A}\mathrm{Y}_\mathrm{A} = \xi^1_\mathcal{M}\mathrm{X}_1 + \ldots \xi^m_\mathcal{M}\mathrm{X}_m + \xi^1\mathrm{Y}_1 + \cdots + \xi^{\dim \mathbf{\Sigma}^n_{m,m-1} - m}\mathrm{Y}_{\dim \mathbf{\Sigma}^n_{m,m-1} - m}. \tag{5.66}$$

Since the exterior ideal does not contain functions, every point of $\boldsymbol{\Sigma}^n_{m,m-1}$ is a 0-integral element. Let us consider then $(\mathrm{E}_0)_z = z \in \boldsymbol{\Sigma}^n_{m,m-1}$. The polar space of E_0 is defined as follows: $\mathrm{H}(\mathrm{E}_0) = \{\xi \in \mathrm{T}_z\boldsymbol{\Sigma}^n_{m,m-1} | \varpi^{\sigma\binom{i}{j}}(\xi) = 0\}$. Every vector ξ satisfying $\xi^{\sigma\binom{i}{j}} = 0$ belongs to the polar space of E_0. Therefore, $\mathrm{C}_0 = n(n-1)/2$. Let us consider then $e_1 = \mathrm{X}_1 + \alpha_1^{\mathrm{A}}\mathrm{Y}_{\mathrm{A}}$, where $\mathrm{A} = n(n+1)/2 + n(n-1)/2 + 1, \ldots, \dim\boldsymbol{\Sigma}^n_{m,m-1} - m$. Let $\mathrm{E}_1 = (z, e_1)$ be a 1-integral element of $\mathcal{I}^n_{m,m-1}$.

The polar space of E_1 is: $\mathrm{H}(\mathrm{E}_1) = \{\xi \in \mathrm{T}_z\boldsymbol{\Sigma}^n_{m,m-1} | \varpi^{\sigma\binom{i}{j}}(\xi) = (\varpi^{\sigma\binom{a}{i}} \wedge \varpi^{\sigma\binom{a}{j}} - \Omega^i_j)(\xi, e_1) = 0\}$, where

$$(\sum_a \varpi^{\sigma\binom{a}{i}} \wedge \varpi^{\sigma\binom{a}{j}} - \Omega^i_j)(\xi, e_1) = \sum_{a=n+1}^{n+\kappa^n_{m,m-1}} \left(\alpha_1^{\sigma\binom{a}{j}} \xi^{\sigma\binom{a}{i}} - \alpha_1^{\sigma\binom{a}{i}} \xi^{\sigma\binom{a}{j}}\right) + \sum_{\mu=2}^{m} \mathcal{R}^i_{j,1\mu} \xi^{\mu}_{\mathcal{M}} = 0. \quad (5.67)$$

Hence, $\mathrm{C}_1 = n(n-1)/2 + n(n-1)/2$. Let us consider $e_2 = \mathrm{X}_2 + \alpha_2^{\mathrm{A}}\mathrm{Y}_{\mathrm{A}}$ such that the coefficients α_2^{A} satisfy :

$$\sum_{a=n+1}^{n+\kappa^n_{m,m-1}} \left(\alpha_1^{\sigma\binom{a}{j}} \alpha_2^{\sigma\binom{a}{i}} - \alpha_1^{\sigma\binom{a}{i}} \alpha_2^{\sigma\binom{a}{j}}\right) = -\mathcal{R}^i_{j,12}. \quad (5.68)$$

The polar space of E_λ, where $\lambda = 1, \ldots, m-2$ is: $\mathrm{H}(\mathrm{E}_\lambda)_{\lambda=1,\ldots,m-2} = \{\xi \in \mathrm{T}_z\boldsymbol{\Sigma}^{\mathbf{n}}_{\mathbf{m;m-1}} | \varpi^{\sigma\binom{i}{j}}(\xi) = (\varpi^{\sigma\binom{a}{i}} \wedge \varpi^{\sigma\binom{a}{j}} - \Omega^i_j)(\xi, e_\mu)_{\mu=1,\ldots,\lambda} = 0\}$. The polar system is then:

$$\begin{cases} \xi^{\sigma\binom{i}{j}} = 0 \\ \sum \left(\alpha_\mu^{\sigma\binom{a}{j}} \xi_\lambda^{\sigma\binom{a}{i}} - \alpha_\mu^{\sigma\binom{a}{i}} \xi_\lambda^{\sigma\binom{a}{j}}\right) = -\sum_{\mu\neq\lambda} \mathcal{R}^i_{j,\gamma\lambda} \xi^{\gamma}_{\mathcal{M}} \quad \text{for } \mu = 1, \ldots m-2 \end{cases} \quad (5.69)$$

Therefore, $\mathrm{C}_\lambda = n(n-1)/2 + \lambda n(n-1)/2$. Let us consider then $e_\lambda = \mathrm{X}_\lambda + \alpha_\lambda^{\mathrm{A}}\mathrm{Y}_{\mathrm{A}}$ such that the coefficients $\alpha_\lambda^{\mathrm{A}}$ are solutions to the following system:

$$\sum_{a=n+1}^{\mathrm{N}} \left(\alpha_\mu^{\sigma\binom{a}{j}} \alpha_\lambda^{\sigma\binom{a}{i}} - \alpha_\mu^{\sigma\binom{a}{i}} \alpha_\lambda^{\sigma\binom{a}{j}}\right) = -\mathcal{R}^i_{j,\mu\lambda} \quad \text{for } \mu = 1, \ldots m-2 \quad (5.70)$$

Finally, the polar space of E_{m-1} is: $\mathrm{H}(\mathrm{E}_{m-1}) = \{\xi \in \mathrm{T}_z\boldsymbol{\Sigma}^{\mathbf{n}}_{\mathbf{m,m-1}} | \varpi^{\sigma\binom{i}{j}}(\xi) = \Big(\sum_a \varpi^{\sigma\binom{a}{i}} \wedge$

$\varpi^{\sigma\binom{a}{j}} - \Omega^i_j\Big)(\xi, e_\lambda)_{\lambda=1,\dots m-1} = (\varpi^{\sigma\binom{a}{i}} \wedge \phi^i)(\xi, e_1, \dots, e_{m-1}) = 0\}.$

$$
\begin{aligned}
&(\varpi^{\sigma\binom{a}{i}} \wedge \phi^i)(\xi, e_1, \dots, e_{m-1}) = \\
&\sum_{i=1}^{n} \Big(\sum_{\lambda=1}^{m} (-1)^{\lambda+1} \psi^i_{\Lambda\smallsetminus\lambda} (\eta^1 \wedge \dots \wedge \eta^{\lambda-1} \wedge \varpi^{\sigma\binom{a}{i}} \wedge \eta^{\lambda+1} \wedge \dots \wedge \eta^m) \Big)(e_1, \dots, e_{m-1}, \xi) \\
&= \sum_{i=1}^{n} \sum_{\lambda=1}^{m} (-1)^{\lambda+1} \psi^i_{\Lambda\smallsetminus\lambda}
\begin{vmatrix}
\xi^1_{\mathcal{M}} & 1 & 0 & \cdots & \cdots & \cdots & 0 \\
\vdots & 0 & \ddots & & & & \vdots \\
\xi^{k-1}_{\mathcal{M}} & 0 & \cdots & 0 & 1 & 0 \cdots & 0 \\
\xi^{\sigma\binom{a}{i}} & \alpha^{\sigma\binom{a}{i}}_1 & \cdots & \alpha^{\sigma\binom{a}{i}}_{k-1} & \alpha^{\sigma\binom{a}{i}}_k & \alpha^{\sigma\binom{a}{i}}_{k+1} \cdots & \alpha^{\sigma\binom{a}{i}}_{m-1} \\
\xi^{k+1}_{\mathcal{M}} & 0 & \cdots & 0 & 1 & 0 \cdots & 0 \\
\vdots & \vdots & & & & \ddots & 0 \\
\vdots & 0 & \cdots & \cdots & \cdots & 0 & 1 \\
\xi^m_{\mathcal{M}} & 0 & \cdots & \cdots & \cdots & \cdots & 0
\end{vmatrix} \\
&= \sum_{i=1}^{n} \Big(-(-1)^{m+1} \sum_{\lambda=1}^{m-1} \alpha^{\sigma\binom{a}{i}}_\lambda \psi^i_{\Lambda\smallsetminus m} \xi^\lambda_{\mathcal{M}} + \sum_{\lambda=1}^{m-1} (-1)^{\lambda+1} \alpha^{\sigma\binom{a}{i}}_\lambda \psi^i_{\Lambda\smallsetminus\lambda} \xi^m_{\mathcal{M}} + (-1)^{m+1} \psi^i_{\Lambda\smallsetminus m} \xi^{\sigma\binom{a}{i}} \Big)
\end{aligned}
\tag{5.71}
$$

Therefore $\mathrm{C}_{m-1} = n(n-1)/2 + (m-1)n(n-1)/2 + \kappa^n_{m,m-1}$. Let us consider $e_m = \mathrm{X}_m + \alpha^{\mathrm{A}}_m \mathrm{Y}_{\mathrm{A}}$. The coefficients α^{A}_m are chosen such that the following system admits a solution

$$
\begin{cases}
\sum_a \Big(\alpha^{\sigma\binom{a}{j}}_\mu \alpha^{\sigma\binom{a}{i}}_\lambda - \alpha^{\sigma\binom{a}{i}}_\mu \alpha^{\sigma\binom{a}{j}}_\lambda \Big) = \mathcal{R}^i_{j,\mu\lambda} \quad \text{where } \mu = 1, \dots m-1 \\
\sum_{i=1}^{n} \psi^i_{\Lambda\smallsetminus m} \xi^{\sigma\binom{a}{i}} = \sum_{i=1}^{n} \sum_{\lambda=1}^{m-1} (-1)^{m+\lambda} \alpha^{\sigma\binom{a}{i}}_\lambda \psi^i_{\Lambda\smallsetminus m}
\end{cases}
\tag{5.72}
$$

$$
\begin{aligned}
\sum_{\lambda=0}^{m-1} \mathrm{C}_\lambda &= \frac{n(n-1)}{2} + (m-2)\Big(\frac{n(n-1)}{2} + \frac{n(n-1)}{2}\Big) + \frac{n(n-1)}{2} + (m-1)\frac{n(n-1)}{2} + \kappa^n_{m,m-1} \\
&= m.\frac{n(n-1)}{2} + \frac{n(n-1)}{2}\frac{m(m-1)}{2} + \kappa^n_{m,m-1}.
\end{aligned}
\tag{5.73}
$$

The coefficients α^{A}_i are actually the coefficients $\mathrm{H}^a_{i\lambda}$ provided by the lemma 5.14, which assures the existence of solutions to the successive polar systems during the construction of the integral flag. The coefficients $\xi^{\sigma\binom{i}{j}}$ for all $1 \leqslant i < j \leqslant n$ are zero for all of the vectors e because of $\varpi^{\sigma\binom{i}{j}}$. Let us denote $\mathrm{E}_\lambda = \mathrm{span}\{e_1, \dots, e_\lambda\}$. The integral flag is then $\mathrm{F} = \mathrm{E}_0 \subset \mathrm{E}_1 \subset \dots \subset \mathrm{E}_{m-1} \subset \mathrm{E}_m$. The Cartan characters are the same as computed previously and the Cartan test assures that the flag is ordinary. By construction, the flag does not annihilate the volume form η^Λ.

5.A The case $(\wedge^2 \mathrm{T}\mathcal{M}^3, g, \mathcal{M}^3, \nabla, \mathrm{Id}_{\wedge^2\mathrm{T}\mathcal{M}^3})$

We explicit in this subappendix the case $(\wedge^2 \mathrm{T}\mathcal{M}^3, g, \mathcal{M}^3, \nabla, \mathrm{Id}_{\wedge^2\mathrm{T}\mathcal{M}^3})$ to better understand the equations, identities and constructions of theorem 4.12 proof, and also to introduce the next chapter.

$(\wedge^\lambda \mathrm{T}\mathcal{M}^m, \mathcal{M}^m, g, \nabla, \mathrm{Id}_{\wedge^\lambda\mathrm{T}\mathcal{M}^m})$ is a special case of vector bundle of rank $m!/\lambda!(m-\lambda)!$ over an m-dimensional manifold $\mathcal{M}^m$ endowed with a vector bundle valued differential λ-form.

Consider over a 3-dimensional manifold $\mathcal{M}^3$ the 3-rank vector bundle $\wedge^2\mathrm{T}\mathcal{M}^3$ consisting of the contravariant 2-tensors of $\mathrm{T}\mathcal{M}^3$. Let $\phi \in \Gamma(\wedge^2\mathrm{T}\mathcal{M}^3 \otimes \wedge^2\mathrm{T}^*\mathcal{M}^3)$ be a covariantly closed $\wedge^2\mathrm{T}\mathcal{M}^3$-valued differential 2-form on $\mathcal{M}^3$ where $\phi = \mathrm{Id}_{\wedge^2\mathrm{T}\mathcal{M}^3}$. Let g be a metric bundle and ∇ be a g-connection on $\wedge^2\mathrm{T}\mathcal{M}^3$. Consider $(\mathrm{E}_1, \mathrm{E}_2, \mathrm{E}_3)$ a g-orthonormal frame on $\wedge^2\mathrm{T}\mathcal{M}^3$ and denote by $(\eta^1\wedge\eta^2, \eta^1\wedge\eta^3, \eta^2\wedge\eta^3)$ the associated coframe. Locally, $\phi = \mathrm{E}_i\phi^i = \mathrm{Id}_{\wedge^2\mathrm{T}\mathcal{M}^3}$ and hence

$$\phi^1 = \eta^1\wedge\eta^2, \phi^2 = \eta^1\wedge\eta^3 \text{ and } \phi^3 = \eta^2\wedge\eta^3. \tag{5.74}$$

The generalized isometric embedding problem, in this case, is equivalent to finding 3-integral manifolds of

$$\mathcal{I}^\wedge_{3,2} = \{\omega^1_2-\eta^1_2, \omega^1_3-\eta^1_3, \omega^2_3-\eta^2_3, \omega^1_a\wedge\omega^a_2+\Omega^1_2, \omega^1_a\wedge\omega^a_3+\Omega^1_3, \omega^2_a\wedge\omega^a_3+\Omega^2_3, \omega^a_1\wedge\phi^1+\omega^a_2\wedge\phi^2+\omega^a_3\wedge\phi^3\} \tag{5.75}$$

on the product manifold

$$\mathbf{\Sigma}^\wedge_{\mathbf{3,2}} = \mathcal{M}^3 \times \mathrm{SO}(3+\kappa^\wedge_{3,2})/\mathrm{SO}(\kappa^\wedge_{3,2}). \tag{5.76}$$

We summarize the following results:

- **Generalized Bianchi identities** are trivial and hence the generalized curvature space in a given point of $\mathcal{M}^3$ is $\mathcal{K}^\wedge_{3,2} = \wedge^2\mathbb{R}^3 \otimes \wedge^2\mathbb{R}^3$.
- **Generalized Cartan identities:** for each normal direction $a = 4, \ldots, 3+\kappa^\wedge_{3,2}$, we have the following identity:
$$\mathrm{H}^a_{31} - \mathrm{H}^a_{22} + \mathrm{H}^a_{13} = 0. \tag{5.77}$$
- **Generalized Gauss map** is $\mathrm{H}_{i\lambda}\mathrm{H}_{j\mu} - \mathrm{H}_{j\lambda}\mathrm{H}_{i\mu} = \mathcal{R}^i_{j;\lambda\mu}$.
- **The submertivity of the generalized Gauss map** The flag of the proposition 5.18 in the proof of Lemma 5.14 is
$$0 = \mathcal{E}_3|^\wedge_{3,2} \subset (\mathcal{E}_2|^\wedge_{3,2} \cap \mathcal{E}^2_3|^\wedge_{3,2}) \subset \mathcal{E}_2|^\wedge_{3,2} \subset \mathcal{E}^2_2|^\wedge_{3,2} \subset \mathcal{E}_1|^\wedge_{3,2} = \mathcal{K}^\wedge_{3,2}. \tag{5.78}$$

We extract a submatrix from $\mathrm{d}\mathcal{G}^\wedge_{3,2}$, where some columns are as follows:

$$\begin{aligned}
\mathrm{d}\mathcal{G}^\wedge_{3,2}(\partial/\partial\mathrm{H}^a_{22}) &= \Big((\mathbf{H^a_{11}}\varepsilon^{\mathbf{1}}_{\mathbf{2;12}}) - \mathrm{H}^a_{31}(\varepsilon^2_{3;12}+\varepsilon^1_{3;13}) - \mathrm{H}^a_{21}\varepsilon^1_{2;13} + (2\mathrm{H}^a_{22}-\mathrm{H}^a_{31})\varepsilon^1_{2;23} \\
&\quad -\mathrm{H}^a_{32}\varepsilon^1_{3;23} + \mathrm{H}^a_{33}\varepsilon^2_{3;23}\Big) \in \mathcal{K}^\wedge|_{3,2} \\
\mathrm{d}\mathcal{G}^\wedge_{3,2}(\partial/\partial\mathrm{H}^a_{32}) &= \Big((\mathbf{H^a_{11}}\varepsilon^{\mathbf{1}}_{\mathbf{3;12}} + \mathbf{H_{21}}\varepsilon^{\mathbf{2}}_{\mathbf{3;12}}) + (\mathrm{H}^a_{31}-\mathrm{H}_{22})\varepsilon^1_{3;23} - \mathrm{H}^a_{23}\varepsilon^2_{3;23}\Big) \in \mathcal{E}^2_2|^\wedge_{3,2} \\
\mathrm{d}\mathcal{G}^\wedge_{3,2}(\partial/\partial\mathrm{H}^a_{23}) &= \Big((\mathbf{H^a_{11}}\varepsilon^{\mathbf{1}}_{\mathbf{2;13}} + \mathbf{H_{12}}\varepsilon^{\mathbf{1}}_{\mathbf{2;23}}) - \mathrm{H}^a_{31}\varepsilon^2_{3;13} - \mathrm{H}^a_{23}\varepsilon^2_{3;23}\Big) \in \mathcal{E}_2|^\wedge_{3,2} \\
\mathrm{d}\mathcal{G}^\wedge_{3,2}(\partial/\partial\mathrm{H}^a_{33}) &= \Big((\mathbf{H^a_{11}}\varepsilon^{\mathbf{1}}_{\mathbf{3;13}} + \mathbf{H_{12}}\varepsilon^{\mathbf{1}}_{\mathbf{3;23}} + \mathbf{H^a_{21}}\varepsilon^{\mathbf{2}}_{\mathbf{3;13}} + \mathbf{H^a_{22}}\varepsilon^{\mathbf{2}}_{\mathbf{3;23}})\Big) \in (\mathcal{E}^2_2|^\wedge_{3,2} \cap \mathcal{E}^2_3|^\wedge_{3,2}).
\end{aligned}$$

Thus, the matrix $\Big((\mathrm{d}\mathcal{G}^{\wedge}_{3,2}(\partial/\partial \mathrm{H}^a_{22}))_a, (\mathrm{d}\mathcal{G}^{\wedge}_{3,2}(\partial/\partial \mathrm{H}^a_{32}))_a, (\mathrm{d}\mathcal{G}^{\wedge}_{3,2}(\partial/\partial \mathrm{H}^a_{23}))_a, (\mathrm{d}\mathcal{G}^{\wedge}_{3,2}(\partial/\partial \mathrm{H}^a_{33})_a\Big)$ is triangular by different sized blocks, and if $\kappa^{\wedge}_{3,2} \geqslant 4$, $\mathrm{d}\mathcal{G}^{\wedge}_{3,2}$ is of maximal rank.

$$\begin{pmatrix} \mathrm{H}^a_{11} & 0 & 0 & 0 \\ 0 & \mathrm{H}^a_{11} & 0 & 0 \\ * & \mathrm{H}^a_{21} & 0 & 0 \\ * & 0 & \mathrm{H}^a_{11} & 0 \\ * & 0 & \mathrm{H}^a_{12} & 0 \\ * & 0 & 0 & \mathrm{H}^a_{11} \\ * & * & 0 & \mathrm{H}^a_{12} \\ 0 & 0 & * & \mathrm{H}^a_{21} \\ * & * & * & \mathrm{H}^a_{22} \end{pmatrix} \tag{5.79}$$

Chapter 6

Other generalized isometric embedding results

In the last chapter, we investigate the generalized isometric embedding in the case of a vector bundle $\mathbb{V}^3$ of rank 3 over a 4-dimensional manifold $\mathcal{M}^4$, endowed with a metric g, an anti-self-dual connection ∇, and a covariantly closed vector bundle valued differential 2-form ϕ. We will use the results of chapter 5, where a general strategy for the proof is expounded. The notations remain the same. For a warmup, expounded in section 1 is a positive answer to the case of covariantly closed vector bundle valued differential 1-forms.

6.1 Covariantly closed differential 1-forms

The following theorem is a positive answer for the generalized isometric embedding problem in the case of the 2-rank vector bundle $\mathbb{V}^2$, an m-dimensional manifold $\mathcal{M}^m$ and a closed covariant non-degenerate[1]$\mathbb{V}^2$-valued differential 1-form ϕ. The result can easily be generalized for any rank bundle, i.e., n is arbitrary and is explained in [Hél09].

Theorem 6.1 – $(\mathbb{V}^2, \mathcal{M}^m, g, \nabla, \phi)_1$ case Let $\mathbb{V}^2$ be a real analytic 2-dimensional vector bundle over a real analytic m-dimensional manifold $\mathcal{M}$ endowed with a metric g and a connection ∇ compatible with g. Given a non-vanishing covariantly closed non-degenerate $\mathbb{V}$-valued differential 1-form ϕ, there exists a local isometric embedding of $\mathbb{V}^2$ in $\mathcal{M} \times \mathbb{R}^{n+\kappa^2_{m,1}}$ over $\mathcal{M}$ where $\kappa^n_{m,m-1} \geqslant 1$ such that the image of ϕ is a conservation law.

We proceed gradually to prove this result: First, when $m = 3$ and for a special ϕ. Than when m is arbitrary with the same special ϕ. Finally, we will explain when ϕ is arbitrary.

Consider the case of a 2-rank vector bundle $\mathbb{V}^2$ over a 3-dimensional manifold $\mathcal{M}^3$ endowed with a metric g, a g-compatible connection ∇ and a covariantly closed $\mathbb{V}^2$-valued differential 1-form ϕ. We use the same notations introduced in the general strategy as expounded in chapter 5. The generalized isometric embedding problem is equivalent to finding integral manifolds of

$\mathcal{I}^2_{3,1} = \{\omega^1_2 - \eta^1_2, \omega^1_a \wedge \omega^a_2 + \Omega^1_2, \omega^a_1 \wedge \eta^1 + \omega^a_2 \wedge \phi^2\}$ on the manifold $\mathbf{\Sigma^2_{3,1}} = \mathcal{M}^3 \times \mathrm{SO}(2+\kappa^2_{3,1})/\mathrm{SO}(\kappa^2_{3,1})$.

The generalized Bianchi identities are: $\Omega^1_2 \wedge \phi^1 = \Omega^1_2 \wedge \phi^1 = 0$, where $\Omega^1_2 = \mathcal{R}^1_{2;12}\eta^{12} + \mathcal{R}^1_{2;13}\eta^{13} + \mathcal{R}^1_{2;23}\eta^{23}$ and $\phi^i = \psi^i_j\eta^j = \psi^i_1\eta^1 + \psi^i_2\eta^2 + \psi^i_3\eta^3$. Consequently, the generalized Bianchi identities can be expressed by the following system:

$$\begin{cases} \mathcal{R}^1_{2;12}\psi^2_3 - \mathcal{R}^1_{2;13}\psi^2_2 + \mathcal{R}^1_{2;23}\psi^2_1 = 0 \\ \mathcal{R}^1_{2;12}\psi^1_3 - \mathcal{R}^1_{2;13}\psi^1_2 + \mathcal{R}^1_{2;23}\psi^1_1 = 0 \end{cases} \tag{6.1}$$

[1] $\phi = \mathrm{E}_i\phi^i = \mathrm{E}_i\psi^i\lambda\eta^\lambda$ is non-degenerate means that the matrix ψ^i_λ is of maximal rank.

and the generalized Caratn identities are in that case expressed by the following system:

$$\begin{cases} \mathrm{H}^a_{11}\psi^1_2 + \mathrm{H}^a_{21}\psi^2_2 - \mathrm{H}^a_{12}\psi^1_1 - \mathrm{H}^a_{22}\psi^2_1 = 0 \\ \mathrm{H}^a_{11}\psi^1_3 + \mathrm{H}^a_{21}\psi^2_3 - \mathrm{H}^a_{13}\psi^1_1 - \mathrm{H}^a_{23}\psi^2_1 = 0 \\ \mathrm{H}^a_{12}\psi^1_3 + \mathrm{H}^a_{22}\psi^2_3 - \mathrm{H}^a_{13}\psi^1_2 - \mathrm{H}^a_{23}\psi^2_2 = 0 \end{cases} \tag{6.2}$$

for each normal direction $a = 3, \dots, 2 + \kappa^2_{3,1}$. Consequently, depending on the values of the functions ψ^i_j, the $\dim\mathcal{K}^2_{3,1}$ can be either 1 or 2. If the 2×3 matrix of the functions ψ is of maximal rank, than $\dim\mathcal{K}^2_{3,1} = 1$.

Examples 6.2 — Generalized Bianchi and generalized Cartan identities. If $\phi = \mathrm{E}_1\phi^1 + \mathrm{E}_2\phi^2 = \mathrm{E}_1\eta^1 + \mathrm{E}_2\eta^2$, i.e., $\psi^1_1 = \psi^2_2 = 1$ and $\psi^1_2 = \psi^2_1 = \psi^1_3 = \psi^2_3 = 0$, then the generalized Bianchi identities are $\mathcal{R}^1_{2;13} = \mathcal{R}^1_{2;23} = 0$, and the generalized Cartan identities are $\mathrm{H}_{13} = \mathrm{H}_{23} = 0$ and $\mathrm{H}_{12} = \mathrm{H}_{21}$. More generally, in the case $(\mathbb{V}^2, \mathcal{M}^m, g, \nabla, \phi)_1$, where $\phi = \mathrm{E}_1\eta^1 + \mathrm{E}_2\eta^2$, the generalized Bianchi identities assure that $\mathcal{R}^1_{2;12}$ is the only non-vanishing term of the curvature tensor and the generalized Cartan identities assure that $\mathrm{H}_{12} = \mathrm{H}_{21}$ and $\mathrm{H}^a_{1\lambda} = \mathrm{H}^a_{2\lambda} = 0$ for all $\lambda = 3, \dots, m$ and for all normal directions.

Let us also consider the special case where the matrix ψ is of maximal rank, for instance $\phi = \mathrm{E}_1\phi^1 + \mathrm{E}_2\phi^2$. This implies that the dimension of the generalized curvature space $\mathcal{K}^2_{3,1}$ is one, i.e., spanned by $\mathcal{R}^1_{2;12}$. Thus, the generalized Gauss equation is then $\mathrm{H}_{11}\mathrm{H}_{22} - \mathrm{H}_{12}.\mathrm{H}_{12} = \mathcal{R}^1_{2;12}$, and is a surjective submersion if $\mathrm{H}_{11} \neq 0$, which is similar to the Gauss equation of surfaces. Consequently, $\kappa^2_{3,1} = 1$. In order to check the involution of the EDS, we need to compute the codimension of $\mathcal{V}_2(\mathcal{I}^2_{3,1}, \eta^1 \wedge \eta^2)$ and the characters C_λ by applying Proposition 2.34 to enumerate the number of linearly independent differential 1-forms $\sum(\mathrm{H}^a_{j\lambda}\pi^a_i - \mathrm{H}^a_{i\lambda}\pi^a_j)$ and $\psi^i_{\lambda_1\dots\lambda_p}\pi^a_i$ that appear in equation 5.14 and 5.15. On one hand,

- There is only one 1-form in the EDS $(\omega^1_2 - \eta^1_2)$, and hence $\mathrm{C}_0 = 1$.
- $(\mathrm{H}^3_{21}\pi^3_1 - \mathrm{H}^3_{11}\pi^3_2)$ and π^3_1 are linearly independent 1-forms and hence $\mathrm{C}_1 = 1 + 2 = 3$.
- There are only two independent 1-forms between the 1-forms $(\mathrm{H}^3_{21}\pi^3_1 - \mathrm{H}^3_{11}\pi^3_2)$, $(\mathrm{H}^3_{22}\pi^3_1 - \mathrm{H}^3_{11}\pi^a_2)$, π^3_1 and π^3_2. Thus, $\mathrm{C}_2 = 1 + 2 = 3$.

On the other hand,

- $\dim\mathbf{\Sigma^2_{3,1}} = 3 + 3 = 6$.
- $\dim\mathrm{G}_2(\mathrm{T}\mathbf{\Sigma^2_{3,1}}, \eta^{\mathbf{1}} \wedge \eta^{\mathbf{2}}) = 15$.
- $\dim\mathcal{H}^2_{3,1} = 2$.
- $\dim\mathcal{Z}^2_{3,1} = 8$.
- Finally, $\operatorname{codim}\mathcal{V}_2(\mathcal{I}^2_{3,1}, \eta^1 \wedge \eta^2) = 7$.

Thus, the exterior differential system passes Cartan's test, and the Cartan–Kähler theorem assures the existence of an integral manifold and hence of a generalized isometric embedding.

For $(\mathbb{V}^2, \mathcal{M}^m, g, \nabla, \phi)_1$, where $\phi = \mathrm{E}_1\phi^1 + \mathrm{E}_2\phi^2$, the same calculation holds. Indeed, $\kappa^2_{m,1} = 1$, and the characters are $\mathrm{C}_0 = 1$, $\mathrm{C}_1 = \cdots = \mathrm{C}_{m-1} = 3$. Thus $\mathrm{C}_0 + \mathrm{C}_1 + \cdots + \mathrm{C}_{m-1} = 3m - 2$.

Besides, $\dim\mathbf{\Sigma^2_{m,1}} = m+3$, $\dim\mathrm{G}_2(\mathrm{T}\mathbf{\Sigma^2_{m,1}}, \boldsymbol{\eta^1} \wedge \boldsymbol{\eta^2}) = 4m+3$, $\dim\mathcal{H}^2_{m,1} = 2$, $\dim\mathcal{Z}^2_{m,1} = m+5$ and finally, $\mathrm{codim}\mathcal{V}_2(\mathcal{I}^2_{3,1}, \eta^1 \wedge \eta^2) = 3m-2$.

Remark 6.3 Consider the $(\mathbb{V}^2, \mathcal{M}^3, g, \nabla, \phi)_1$ case, where $\phi = \mathrm{E}_1(\eta^1 + \psi^1_3\eta^3) + \mathrm{E}_2(\eta^2 + \psi^2_3\eta^3)$. The generalized Bianchi identities (6.1) are: $\psi^2_3\mathcal{R}^1_{2;12} = \mathcal{R}^1_{2;13}$ and $-\psi^1_3\mathcal{R}^1_{2;12} = \mathcal{R}^1_{2;23}$, and the generalized Cartan identities (6.2) are: $\mathrm{H}^a_{12} = \mathrm{H}^a_{21}$, $\mathrm{H}^a_{13} = \psi^1_3\mathrm{H}^a_{11} + \psi^2_3\mathrm{H}^a_{21}$ and $\mathrm{H}^a_{23} = \psi^1_3\mathrm{H}^a_{12}+\psi^2_3\mathrm{H}^a_{22}$. Then if the generalized Cartan identities are substituted in the generalized Gauss equations $\mathrm{H}_{11}\mathrm{H}_{22}-\mathrm{H}_{12}\mathrm{H}_{21} = \mathcal{R}^1_{2;12}$, $\mathrm{H}_{11}\mathrm{H}_{23}-\mathrm{H}_{13}\mathrm{H}_{21} = \mathcal{R}^1_{2;13}$ and $\mathrm{H}_{12}\mathrm{H}_{23}-\mathrm{H}_{13}\mathrm{H}_{22} = \mathcal{R}^1_{2;23}$, then we recover the generalized Bianchi identities. Moreover, the above characters' computations and the codimension of $\mathcal{V}_2(\mathcal{I}^2_{3,1}, \eta^1 \wedge \eta^2)$ are the same, and thus lead to the same conclusion.

6.2 Generalized isometric embedding of 2-form with anti-self dual condition

Consider a vector bundle $\mathbb{V}^3$ of rank 3 over a 4-dimensional manifold. Let $\phi \in \Gamma(\mathbb{V}^3 \otimes \wedge^2\mathrm{T}^*\mathcal{M}^4)$ be a $\mathbb{V}^3$-valued differential 2-form on $\mathcal{M}^4$. Let g be a metric bundle and ∇ be a g-connection on $\mathbb{V}^3$. Denote by $(\mathrm{E}_1, \mathrm{E}_2, \mathrm{E}_3)$ a g-orthonormal frame on $\mathbb{V}^3$, and by $(\eta^1 \wedge \eta^2, \eta^1 \wedge \eta^3, \eta^2 \wedge \eta^3, \eta^1 \wedge \eta^4, \eta^2 \wedge \eta^4, \eta^3 \wedge \eta^4)$ a coframe of $\wedge^2\mathrm{T}^*\mathcal{M}^4$. Let us consider the case of ϕ defined as follows:

$$\phi^1 = \eta^1 \wedge \eta^2 + \eta^3 \wedge \eta^4, \phi^2 = \eta^1 \wedge \eta^3 - \eta^2 \wedge \eta^4 \text{ and } \phi^3 = \eta^2 \wedge \eta^3 + \eta^1 \wedge \eta^4 \tag{6.3}$$

Theorem 6.4 – Generalized isometric embedding of 2-form with ASD condition Let $\mathcal{M}^4$ be a real analytic 4-dimensional manifold. Consider a real analytic vector bundle $\mathbb{V}^3$ of rank 3 over $\mathcal{M}^4$, endowed with a Riemannian metric g, an anti-self-dual g-compatible connection ∇, and a covariantly closed $\mathbb{V}^3$-valued differential 2-form ϕ of the form (6.3). There exists then a generalized isometric embedding Ψ of $\mathbb{V}^3$ into $\mathcal{M}^4 \times \mathbb{R}^{3+\kappa^3_{4,2,\mathrm{ASD}}}$, where $\kappa^3_{4,2,\mathrm{ASD}} \geqslant 4$, such that $\Psi(\phi)$ is a local conservation law.

We summarize in the following the notions and equations needed for solving the general isometric embedding in this case:

- **The generalized isometric embedding EDS** is equivalent to finding integral manifolds of
$$\mathcal{I}^3_{4,2,\mathrm{ASD}} = \{\omega^i_j - \eta^i_j, \omega^i_a \wedge \omega^a_i + \Omega^i_j, \omega^a_i \wedge \phi^i\}$$
on the manifold
$$\mathbf{\Sigma^3_{4,2,ASD}} = \mathcal{M}^4 \times \mathrm{SO}(6+\kappa^3_{4,2,\mathrm{ASD}})/\mathrm{SO}(\kappa^3_{4,2,\mathrm{ASD}}).$$

- **Generalized Bianchi identities** $\Omega^i_j \wedge \phi^j = 0$ for all $i = 1, \ldots, 3$ are:
$$\left\{\begin{array}{rcl} -\mathcal{R}^1_{2;24} + \mathcal{R}^1_{2;13} + \mathcal{R}^1_{3;14} + \mathcal{R}^1_{3;23} & = & 0 \\ \mathcal{R}^1_{2;34} + \mathcal{R}^1_{2;12} - \mathcal{R}^2_{3;14} - \mathcal{R}^2_{3;23} & = & 0 \\ \mathcal{R}^1_{3;34} + \mathcal{R}^1_{3;12} + \mathcal{R}^2_{3;24} - \mathcal{R}^2_{3;13} & = & 0 \end{array}\right. \tag{6.4}$$
Since $\Omega^i_j = \mathcal{R}^i_{j;\lambda\mu}\eta^{\lambda\mu}$, the anti-self-duality condition on the connection ∇, i.e., $*\Omega+\Omega = 0$, implies that
$$\mathcal{R}^i_{j;12} + \mathcal{R}^i_{j;34} = \mathcal{R}^i_{j;13} - \mathcal{R}^i_{j;24} = \mathcal{R}^i_{j;23} + \mathcal{R}^i_{j;14} = 0 \tag{6.5}$$

and thus the generalized Bianchi identities are trivial. In particular,

$$\dim \mathcal{K}^3_{4,2,\mathrm{ASD}} = 9 \tag{6.6}$$

- **Generalized Cartan identities**, for each normal direction, are given by the following system:

$$\left\{\begin{array}{rcl} \mathrm{H}^a_{13} & = & \mathrm{H}^a_{22} - \mathrm{H}^a_{31} \\ \mathrm{H}^a_{14} & = & \mathrm{H}^a_{32} + \mathrm{H}^a_{21} \\ \mathrm{H}^a_{24} & = & \mathrm{H}^a_{33} - \mathrm{H}^a_{11} \\ \mathrm{H}^a_{34} & = & -\mathrm{H}^a_{23} - \mathrm{H}^a_{12} \end{array}\right. \tag{6.7}$$

- **Generalized Gauss equation** is $\mathrm{H}_{i\lambda}\mathrm{H}_{j\mu} - \mathrm{H}_{i\mu}\mathrm{H}_{j\lambda} = \mathcal{R}^i_{j;\lambda\mu}$.

The existence of suitable coefficients that satisfy generalized Cartan identities and the generalized Gauss equations, and the minimum required embedding codimension $\kappa^3_{4,2,\mathrm{ASD}}$ is provided by the following:

Lemma 6.5 — Yang-Mills type generalized Gauss map submersitivity Let $\kappa^3_{4,2,\mathrm{ASD}} \geqslant 4$. Let $\mathcal{H}^3_{4,2,\mathrm{ASD}} \subset \mathcal{W}^3_{4,2,\mathrm{ASD}} \otimes \mathbb{R}^n \otimes \mathbb{R}^m$ be the open set consisting of those elements $\mathrm{H} = (\mathrm{H}^a_{i\lambda})$, so that the vectors $\{\mathrm{H}_{11}, \mathrm{H}_{12}, \mathrm{H}_{21}, \mathrm{H}_{22}\}$ are linearly independents as elements of $\mathcal{W}^3_{4,2,\mathrm{ASD}}$ and satisfy generalized Cartan identities. Then $\mathcal{G}^3_{4,2,\mathrm{ASD}} : \mathcal{H}^3_{4,2,\mathrm{ASD}} \longrightarrow \mathcal{K}^3_{4,2,\mathrm{ASD}}$ is a surjective submersion.

The proof of lemma 6.5 is similar to the proof of lemma 5.14. Indeed, the submatrix of $\mathrm{d}\mathcal{G}^3_{4,2,\mathrm{ASD}}$ of maximal rank is:

$$\Big((\mathrm{d}\mathcal{G}^3_{4,2,\mathrm{ASD}}(\partial/\partial \mathrm{H}^a_{22}))_a, (\mathrm{d}\mathcal{G}^3_{4,2,\mathrm{ASD}}(\partial/\partial \mathrm{H}^a_{32}))_a, (\mathrm{d}\mathcal{G}^3_{4,2,\mathrm{ASD}}(\partial/\partial \mathrm{H}^a_{23}))_a, (\mathrm{d}\mathcal{G}^3_{4,2,\mathrm{ASD}}(\partial/\partial \mathrm{H}^a_{33}))_a\Big)$$

If $\kappa^3_{4,2,\mathrm{ASD}} \geqslant 4$, then the partial computations of the codimension $\mathcal{V}_4(\mathcal{I}^3_{4,2,\mathrm{ASD}}, \eta^\Lambda)$ are:

- $\dim \mathcal{H}^3_{4,2,\mathrm{ASD}} = 8\kappa^3_{4,2,\mathrm{ASD}} - 9$.
- $\dim \mathbf{\Sigma^3_{4,2,ASD}} = 3\kappa^3_{4,2,\mathrm{ASD}} + 7$.
- $\dim \mathcal{Z}^3_{4,2,\mathrm{ASD}} = 11\kappa^3_{4,2,\mathrm{ASD}} - 2$.
- $\dim \mathrm{G}_4(\mathrm{T}_{(\mathrm{M},\Upsilon)}\mathbf{\Sigma^3_{4,2,ASD}}) = 15\kappa^3_{4,2,\mathrm{ASD}} + 19$.

Finally,

$$\dim \mathcal{V}_4(\mathcal{I}^3_{4,2,\mathrm{ASD}}, \eta^\Lambda) = 4\kappa^3_{4,2,\mathrm{ASD}} + 21. \tag{6.8}$$

Constructing an ordinary integral flag

The exterior differential ideal $\mathcal{I}^3_{4,2,\mathrm{ASD}}$ is generated by 3 differential 1-forms $\omega^i_j - \eta^i_j$, by 3 differential 2-forms $\omega^i_a \wedge \omega^a_j + \Omega^i_j$ and by $\kappa^3_{4,2,\mathrm{ASD}}$ differential 3-forms $\omega^a_i \wedge \phi^i$. As explained in chapter 5, in order to compute the codimension of the consecutive polar spaces of the integral flag, we consider the following forms: $\sum_a \mathrm{H}^a_{j\lambda}\pi^a_i - \mathrm{H}^a_{i\lambda}\pi^a_j$ which comes from the contribution of the differential 2-forms of the exterior differential ideal, and $\psi^i_{\lambda\mu}\pi^a_i$ which comes from the contribution of the differential 3-forms. Hence,

$$\mathrm{C}_0 = 3 \quad , \quad \mathrm{C}_1 = 6 \quad , \quad \mathrm{C}_2 = 9 + \kappa^3_{4,2,\mathrm{ASD}} \quad \text{and} \quad \mathrm{C}_3 = 3 + 3\kappa^3_{4,2,\mathrm{ASD}}. \tag{6.9}$$

and thus, the exterior differential system passes the Cartan test.

6.A Remarks on generalized Bianchi identities

We present a different way to express the generalized Bianchi identities in terms of the minors of the coefficients of the covariantly closed vector bundle valued differential forms, for the following two cases: $(\mathbb{V}^2, \mathcal{M}^3, g, \nabla, \phi)_1$ and $(\mathbb{V}^3, \mathcal{M}^3, g, \nabla, \phi)_1$.

Definition 6.6 – Inner-cross product of $3 \times n$ matrices Let $\mathrm{A} = (a_{ij})$ and $\mathrm{B} = (b_{ij})$ be two real $3 \times n$-matrices. Denote by $l_i(\mathrm{A})$ and $l_i(\mathrm{B})$ the $1 \times n$-matrix $\begin{pmatrix} a_{i1} & \dots & a_{in} \end{pmatrix}$ and $\begin{pmatrix} b_{i1} & \dots & b_{in} \end{pmatrix}$ respectively. Then the inner-cross product of A and B, denoted by $\mathrm{A}\dot{\times}\mathrm{B}, is$ the 3×1-matrix defined as follows:

$$\mathrm{A}\dot{\times}\mathrm{B} = \begin{pmatrix} l_2(\mathrm{A}).^t l_3(\mathrm{B}) - l_3(\mathrm{A}).^t l_2(\mathrm{B}) \\ -l_1(\mathrm{A}).^t l_3(\mathrm{B}) + l_3(\mathrm{A}).^t l_1(\mathrm{B}) \\ l_1(\mathrm{A}).^t l_2(\mathrm{B}) - l_2(\mathrm{A}).^t l_1(\mathrm{B}) \end{pmatrix}. \tag{6.10}$$

Examples 6.7 – Inner-cross product.

1. When $n = 1$, then the inner-cross product is nothing but the usual cross product of vectors in $\mathbb{R}^3$, i.e.,

$$\mathrm{A}\dot{\times}\mathrm{B} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \dot{\times} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ -a_1 b_3 + a_3 b_1 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} \tag{6.11}$$

2. When $n = 2$, the inner-cross product of A and B in $\mathrm{M}_{3\times 2}(\mathbb{R})$ is

$$\mathrm{A}\dot{\times}\mathrm{B} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \dot{\times} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} = \begin{pmatrix} a_{21}b_{31} + a_{22}b_{32} - a_{31}b_{21} - a_{32}b_{22} \\ -a_{11}b_{31} - a_{12}b_{32} + a_{31}b_{11} + a_{32}b_{12} \\ a_{11}b_{21} + a_{12}b_{22} - a_{21}b_{11} - a_{22}b_{12} \end{pmatrix}. \tag{6.12}$$

In the following, we find an non-exhaustive list of the inner-cross product's properties.

Properties 6.8 – Inner-cross product Let A, B and C be three $3 \times n$ real matrices and λ a real number. Then the inner-cross product is:

1. skew-symmetric: $\mathrm{A}\dot{\times}\mathrm{B} = -\mathrm{B}\dot{\times}\mathrm{A}$, and hence $\mathrm{A}\dot{\times}\mathrm{A} = 0$.
2. Compatible with scalar multiplication: $\lambda.(\mathrm{A}\dot{\times}\mathrm{B}) = (\lambda.\mathrm{A})\dot{\times}\mathrm{B} = \mathrm{A}\dot{\times}(\lambda.\mathrm{B})$.
3. Distributive over addition: $(\mathrm{A} + \mathrm{B})\dot{\times}\mathrm{C} = \mathrm{A}\dot{\times}\mathrm{C} + \mathrm{B}\dot{\times}\mathrm{C}$.

6.A.1 $(\mathbb{V}^2, \mathcal{M}^3, g, \nabla, \phi)_1$

Recall that the covariantly closed $\mathbb{V}_2$-valued differential form ϕ is expressed as follows: $\phi = \mathrm{E}_i\phi^i = \mathrm{E}_i\psi^i_\lambda \wedge \eta^\lambda$, and can be written in a matrix form as:

$$\phi = \begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix} = \begin{pmatrix} \psi^1_1 & \psi^1_2 & \psi^1_3 \\ \psi^2_1 & \psi^2_2 & \psi^2_3 \end{pmatrix} \begin{pmatrix} \eta^1 \\ \eta^2 \\ \eta^3 \end{pmatrix} \tag{6.13}$$

The minors of ψ are denoted by $\Delta^{12}_{12}\psi = \psi^1_1\psi^2_2 - \psi^1_2\psi^2_1$, $\Delta^{12}_{13}\psi = \psi^1_1\psi^2_3 - \psi^1_3\psi^2_1$ and $\Delta^{12}_{23}\psi = \psi^1_2\psi^2_3 - \psi^1_3\psi^2_2$. The generalized Bianchi identities 6.1 are equivalent to

$$\Delta\psi\dot{\times}\mathcal{R}\begin{pmatrix}\Delta_{12}\psi\\ \Delta_{13}\psi\\ \Delta_{23}\psi\end{pmatrix}\times\begin{pmatrix}\mathcal{R}^1_{2;12}\\ \mathcal{R}^1_{2;13}\\ \mathcal{R}^1_{2;23}\end{pmatrix}=\begin{pmatrix}\Delta^{12}_{13}\psi\mathcal{R}^1_{2;23}-\Delta^{12}_{23}\psi\mathcal{R}^1_{2;13}\\ -\Delta^{12}_{12}\psi\mathcal{R}^1_{2;23}+\Delta^{12}_{23}\psi\mathcal{R}^1_{2;12}\\ \Delta^{12}_{12}\psi\mathcal{R}^1_{2;13}-\Delta^{12}_{13}\psi\mathcal{R}^1_{2;12}\end{pmatrix}=0 \tag{6.14}$$

Examples 6.9 – $(\mathbb{V}^2, \mathcal{M}^3, \mathbf{g}, \nabla, \phi)_1$-Continued. When $m = 3$ and $\phi = \mathrm{Id}_{\,\mathrm{span}\{\mathrm{E}_1,\mathrm{E}_2\}}$, the only non-vanishing minor is $\Delta^{12}_{12}\psi = 1$. Thus, (6.14) implies the same conclusion as the examples and the remarks of the first section of this chapter, that is: $\mathcal{R}^1_{2;13} = \mathcal{R}^1_{2;23} = 0$. For $\phi = \mathrm{E}_1(\eta^1 + \psi^1_3\eta^3) + \mathrm{E}_2(\eta^2) + \psi^2_3$, the minors are: $\Delta^{12}_{12}\psi = 1$, $\Delta^{12}_{13}\psi = \psi^2_3$ and $\Delta^{12}_{23}\psi = -\psi^1_3$, and hence (6.14) leads to: $\mathcal{R}^1_{2;23} = -\psi^1_3\mathcal{R}^1_{2;12}$ and $\mathcal{R}^1_{2;13} = \psi^2_3\mathcal{R}^1_{2;12}$.

6.A.2 $(\mathbb{V}^3, \mathcal{M}^3, g, \nabla, \phi)_1$

Consider a 2-rank vector bundle $\mathbb{V}^3$ over a 3-dimensional manifold $\mathcal{M}^3$, endowed with a Riemannian metric g and g-connection ∇, and a covariantly closed $\mathbb{V}^3$-valued differential 1-form ϕ, which is expressed

$$\phi = \mathrm{E}_i\phi^i = \begin{pmatrix}\phi^1\\ \phi^2\\ \phi^3\end{pmatrix} = \begin{pmatrix}\psi^1_1 & \psi^1_2 & \psi^1_3\\ \psi^2_1 & \psi^2_2 & \psi^2_3\\ \psi^3_1 & \psi^3_2 & \psi^3_3\end{pmatrix}\begin{pmatrix}\eta^1\\ \eta^2\\ \eta^3\end{pmatrix}. \tag{6.15}$$

The minors $\Delta^{ij}_{\lambda\mu}\psi = \psi^i_\lambda\psi^j_\mu - \psi^i_\mu\psi^j_\lambda$. Then the generalized Bianchi identities are also expressed as follows:

$$\Delta\psi\dot{\times}\mathcal{R} = \begin{pmatrix}\Delta^{12}_{12}\psi & \Delta^{13}_{12}\psi & \Delta^{23}_{12}\psi\\ \Delta^{12}_{13}\psi & \Delta^{13}_{13}\psi & \Delta^{23}_{13}\psi\\ \Delta^{12}_{23}\psi & \Delta^{13}_{23}\psi & \Delta^{23}_{23}\psi\end{pmatrix}\dot{\times}\begin{pmatrix}\mathcal{R}^1_{2;12} & \mathcal{R}^1_{3;12} & \mathcal{R}^2_{3;12}\\ \mathcal{R}^1_{2;13} & \mathcal{R}^1_{3;13} & \mathcal{R}^2_{3;13}\\ \mathcal{R}^1_{2;23} & \mathcal{R}^1_{3;23} & \mathcal{R}^2_{3;23}\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}. \tag{6.16}$$

Example 6.10 – Isometric embedding of $(\mathcal{M}^3, g)$. Since the covariantly closed $\mathrm{T}\mathcal{M}^3$-valued differential 1-form ϕ in the isometric embedding problem of 3-dimensional Riemannian manifold is the identity on $\mathrm{T}\mathcal{M}^3$, the minors $\Delta^{12}_{12}\psi = \Delta^{13}_{13}\psi = \Delta^{23}_{23}\psi = 1$ and all of the others vanish. Thus, (6.16) leads to: $\mathcal{R}^1_{2;13} = \mathcal{R}^1_{3;12}$, $\mathcal{R}^1_{2;23} = \mathcal{R}^2_{3;12}$ and $\mathcal{R}^1_{3;23} = \mathcal{R}^2_{3;13}$, which are the usual Bianchi identities.

Appendix

Appendix A

Computations and proofs

This appendix is dedicated to proving the results stated in sections 1 and 2 of the first chapter. The exact statement of a result is recalled and is followed by its proof. Even though the author believe that the best way to understand the computations is take a paper and pencil and to do them, the computations are expounded in a detailed way in order to help non-experts better understand these kind of computations.

Theorem 1.3 - The functoriality of the curvature Let ∇ be a connection on a vector bundle $\mathbb{V}$ of rank r over an m-dimensional manifold $\mathcal{M}$. Then, for any f, g and h smooth functions on $\mathcal{M}$, $\mathrm{S} \in \Gamma(\mathrm{E})$ a section of ξ, and $\mathrm{X}, \mathrm{Y} \in \Gamma(\mathrm{T}\mathcal{M})$ two tangent vector fields of $\mathcal{M}$, we have:

$$\mathcal{R}^{\nabla}(f\mathrm{X}, g\mathrm{Y})(h\mathrm{S}) = f.g.h.\mathcal{R}^{\nabla}(\mathrm{X}, \mathrm{Y})\mathrm{S}. \tag{A.1}$$

Proof. By definition of the curvature of a connection,

$$\mathcal{R}(f\mathrm{X}, g\mathrm{Y})(h\mathrm{S}) := \Big([\nabla_{f\mathrm{X}}\ \nabla_{g\mathrm{Y}}] - \nabla_{[f\mathrm{X}\ g\mathrm{Y}]}\Big)(h\mathrm{S}) = \underbrace{\nabla_{f\mathrm{X}}\nabla_{g\mathrm{Y}}(h\mathrm{S})}_{\lhd} - \underbrace{\nabla_{g\mathrm{Y}}\nabla_{f\mathrm{X}}(h\mathrm{S})}_{\rhd} - \underbrace{\nabla_{[f\mathrm{X}\ g\mathrm{Y}]}(h\mathrm{S})}_{\Diamond}.$$

$$\begin{aligned}
(\lhd) &= f\nabla_{\mathrm{X}}\Big(\nabla_{g\mathrm{Y}}(hs)\Big) = f\nabla_{\mathrm{X}}\Big(g\nabla_{\mathrm{Y}}(h\mathrm{S})\Big) = f\nabla_{\mathrm{X}}\Big(g\mathrm{Y}(h)\mathrm{S} + gh\nabla_{\mathrm{Y}}\mathrm{S}\Big)\\
&= f\nabla_{\mathrm{X}}\Big(g\mathrm{Y}(h)\mathrm{S}\Big) + f\nabla_{\mathrm{X}}\Big(gh\nabla_{\mathrm{Y}}\mathrm{S}\Big)\\
&= f\mathrm{X}\Big(g\mathrm{Y}(h)\Big)\mathrm{S} + fg\mathrm{Y}(h)\nabla_{\mathrm{X}}\mathrm{S} + f\mathrm{X}(gh)\nabla_{\mathrm{Y}}\mathrm{S} + fgh\nabla_{\mathrm{X}}\nabla_{\mathrm{Y}}\mathrm{S}\\
&= f\mathrm{X}(g)\mathrm{Y}(h)\mathrm{S} + fg\mathrm{X}\Big(\mathrm{Y}(h)\Big)\mathrm{S} + fg\mathrm{Y}(h)\nabla_{\mathrm{X}}\mathrm{S} + fh\mathrm{X}(g)\nabla_{\mathrm{Y}}\mathrm{S} + fg\mathrm{X}(h)\nabla_{\mathrm{Y}}\mathrm{S} + fgh\nabla_{\mathrm{X}}\nabla_{\mathrm{Y}}\mathrm{S}
\end{aligned}$$

The term $(\rhd)$ is obtained without computation, just by interchanging f by g and X by Y. We then have

$$(\rhd) = g\mathrm{Y}(f)\mathrm{X}(h)\mathrm{S} + gf\mathrm{Y}\Big(\mathrm{X}(h)\Big)\mathrm{S} + gf\mathrm{X}(h)\nabla_{\mathrm{Y}}\mathrm{S} + gh\mathrm{Y}(f)\nabla_{\mathrm{X}}\mathrm{S} + gf\mathrm{Y}(h)\nabla_{\mathrm{Y}}\mathrm{S} + gfh\nabla_{\mathrm{Y}}\nabla_{\mathrm{X}}\mathrm{S}$$

To have the last term $(\Diamond)$, let us first compute the Lie brackets $[f\mathrm{X}\ g\mathrm{Y}]$.

$$[f\mathrm{X}\ g\mathrm{Y}] = f\mathrm{X}(g\mathrm{Y}) - g\mathrm{Y}(f\mathrm{X}) = f\mathrm{X}(g)\mathrm{Y} + fg\mathrm{X}(\mathrm{Y}) - g\mathrm{Y}(f)\mathrm{X} - gf\mathrm{Y}(\mathrm{X})$$

and hence,

$$\begin{aligned}
(\Diamond) &= \nabla_{f\mathrm{X}(g)\mathrm{Y}}(h\mathrm{S}) + \nabla_{fg\mathrm{X}(\mathrm{Y})}(h\mathrm{S}) - \nabla_{g\mathrm{Y}(f)\mathrm{X}}(h\mathrm{S}) - \nabla_{gf\mathrm{Y}(\mathrm{X})}(h\mathrm{S})\\
&= f\mathrm{X}(g)\nabla_{\mathrm{Y}}(h\mathrm{S}) + fg\nabla_{\mathrm{X}(\mathrm{Y})}(h\mathrm{S}) - g\mathrm{Y}(f)\nabla_{\mathrm{X}}(h\mathrm{S}) - gf\nabla_{\mathrm{Y}(\mathrm{X})}(h\mathrm{S})\\
&= f\mathrm{X}(g)\mathrm{Y}(h)\mathrm{S} + f\mathrm{X}(g)h\nabla_{\mathrm{Y}}\mathrm{S} + fg\mathrm{X}\Big(\mathrm{Y}(h)\Big)\mathrm{S} + fgh\nabla_{\mathrm{X}(\mathrm{Y})} - g\mathrm{Y}(f)\mathrm{X}(h)\mathrm{S} - g\mathrm{Y}(f)h\nabla_{\mathrm{X}}\\
&\quad - gf\mathrm{Y}\Big(\mathrm{X}(h)\Big)\mathrm{S} - gfh\nabla_{\mathrm{Y}(\mathrm{X})}\mathrm{S}
\end{aligned}$$

Finally, $(\lhd) - (\rhd) - (\Diamond)$ gives

$$\begin{aligned}\mathcal{R}(f\mathrm{X}, g\mathrm{Y})(h\mathrm{S}) &= f\mathrm{X}(g)\mathrm{Y}(h)\mathrm{S} + fg\mathrm{X}\Big(\mathrm{Y}(h)\Big)\mathrm{S} + fg\mathrm{Y}(h)\nabla_{\mathrm{X}}\mathrm{S} + fh\mathrm{X}(g)\nabla_{\mathrm{Y}}\mathrm{S} + fg\mathrm{X}(h)\nabla_{\mathrm{Y}}\mathrm{S}\\ &+ \underbrace{fgh\nabla_{\mathrm{X}}\nabla_{\mathrm{Y}}\mathrm{S}}_{*} - g\mathrm{Y}(f)\mathrm{X}(h)\mathrm{S} - gf\mathrm{Y}\Big(\mathrm{X}(h)\Big)\mathrm{S} - gf\mathrm{X}(h)\nabla_{\mathrm{Y}}\mathrm{S} - gh\mathrm{Y}(f)\nabla_{\mathrm{X}}\mathrm{S} - gf\mathrm{Y}(h)\nabla_{\mathrm{Y}}\mathrm{S}\\ &- \underbrace{gfh\nabla_{\mathrm{Y}}\nabla_{\mathrm{X}}\mathrm{S}}_{*} - f\mathrm{X}(g)\mathrm{Y}(h)\mathrm{S} - f\mathrm{X}(g)h\nabla_{\mathrm{Y}}\mathrm{S} - fg\mathrm{X}\Big(\mathrm{Y}(h)\Big)\mathrm{S} - \underbrace{fgh\nabla_{\mathrm{X}(\mathrm{Y})}\mathrm{S}}_{*} + g\mathrm{Y}(f)\mathrm{X}(h)\mathrm{S}\\ &+ g\mathrm{Y}(f)h\nabla_{\mathrm{X}}\mathrm{S} + gf\mathrm{Y}\Big(\mathrm{X}(h)\Big)\mathrm{S} + \underbrace{gfh\nabla_{\mathrm{Y}(\mathrm{X})}\mathrm{S}}_{*}\end{aligned}$$

Finally, $\mathcal{R}(f\mathrm{X}, g\mathrm{Y})(h\mathrm{S}) = fgh\Big(\nabla_{\mathrm{X}}\nabla_{\mathrm{Y}} - \nabla_{\mathrm{Y}}\nabla_{\mathrm{X}} - \nabla_{\mathrm{X}(\mathrm{Y})} + \nabla_{\mathrm{Y}(\mathrm{X})}\Big)\mathrm{S} = fgh\Big(\nabla_{\mathrm{X}}\nabla_{\mathrm{Y}} - \nabla_{\mathrm{Y}}\nabla_{\mathrm{X}} - \nabla_{[\mathrm{X}\ \mathrm{Y}]}\Big)\mathrm{S} = fgh\mathcal{R}(\mathrm{X}, \mathrm{Y})\mathrm{S}$. All the others terms are pairwise cancelled. □

Theorem 1.7 - Cartan's second-structure equation Let ∇ be a connection on a vector bundle $(\mathbb{V}, \pi, \mathcal{M})$ of rank r over an m-dimensional manifold. Denote by $\omega = (\omega^i_j)$ the $\mathfrak{gl}(r;\mathbb{R})$ valued differential 1-form of the connection ∇. Then

$$\mathrm{d}\omega^i_j + \omega^i_k \wedge \omega^k_j = \Omega^i_j \quad \text{for all } i, j. \tag{A.2}$$

Proof. By definition, $\mathcal{R}(\mathrm{X}, \mathrm{Y})\mathrm{S}_j = \Omega^i_j(\mathrm{X}, \mathrm{Y})\mathrm{S}_i$. In the other hand

$$\mathcal{R}(\mathrm{X}, \mathrm{Y})\mathrm{S}_j = \underbrace{\nabla_{\mathrm{X}}\nabla_{\mathrm{Y}}\mathrm{S}_j}_{\lhd} - \underbrace{\nabla_{\mathrm{Y}}\nabla_{\mathrm{X}}\mathrm{S}_j}_{\rhd} - \underbrace{\nabla_{[\mathrm{X}\ \mathrm{Y}]}\mathrm{S}_j}_{\Diamond}.$$

Hence,

$$\begin{aligned}(\lhd) = \nabla_{\mathrm{X}}\Big(\omega^i_j(\mathrm{Y})\mathrm{S}_i\Big) &= \nabla_{\mathrm{X}}\Big(\omega^i_j(\mathrm{Y})\mathrm{S}_i\Big) = \mathrm{X}\Big(\omega^i_j(\mathrm{Y})\Big)\mathrm{S}_i + \omega^i_j(\mathrm{Y})\nabla_{\mathrm{X}}\mathrm{S}_i\\ &= \mathrm{X}\Big(\omega^i_j(\mathrm{Y})\Big)\mathrm{S}_i + \omega^i_j(\mathrm{Y})\omega^k_i(\mathrm{X})\mathrm{S}_k = \mathrm{X}\Big(\omega^i_j(\mathrm{Y})\Big)\mathrm{S}_i + \omega^i_j(\mathrm{Y})\omega^k_i(\mathrm{X})\mathrm{S}_k\\ &= \mathrm{X}\Big(\omega^i_j(\mathrm{Y})\Big)\mathrm{S}_i + \omega^k_j(\mathrm{Y})\omega^i_k(\mathrm{X})\mathrm{S}_i = \mathrm{X}\Big(\omega^i_j(\mathrm{Y})\Big)\mathrm{S}_i + \omega^i_k(\mathrm{X})\omega^k_j(\mathrm{Y})\mathrm{S}_i.\end{aligned}$$

The term $(\rhd)$ is obtained without computation by interchanging X by Y, and hence $(\rhd) := \mathrm{Y}\Big(\omega^i_j(\mathrm{X})\Big)\mathrm{S}_i + \omega^i_k(\mathrm{Y})\omega^k_j(\mathrm{X})\mathrm{S}_i$. Finally, $(\Diamond) = \omega^i_j([\mathrm{X}\ \mathrm{Y}])\mathrm{S}_i$. Nevertheless, from the Cartan formula

$$(\triangle) : \mathrm{d}\omega^i_j(\mathrm{X}, \mathrm{Y}) = \mathrm{X}\omega^i_j(\mathrm{Y}) - \mathrm{Y}\omega^i_j(\mathrm{X}) - \omega^i_j([\mathrm{X}\ \mathrm{Y}])$$

and

$$(\triangledown) : \omega^i_k \wedge \omega^k_j(\mathrm{X}, \mathrm{Y}) = \begin{vmatrix}\omega^i_k(\mathrm{X}) & \omega^i_k(\mathrm{Y})\\ \omega^k_j(\mathrm{X}) & \omega^k_j(\mathrm{Y})\end{vmatrix} = \omega^i_k(\mathrm{X})\omega^k_j(\mathrm{Y}) - \omega^i_k(\mathrm{Y})\omega^k_j(\mathrm{X})$$

We conclude that

$$\begin{aligned}\mathcal{R}(\mathrm{X}, \mathrm{Y})\mathrm{S}_j &= \mathrm{X}\Big(\omega^i_j(\mathrm{Y})\Big)\mathrm{S}_i + \omega^i_k(\mathrm{X})\omega^k_j(\mathrm{Y})\mathrm{S}_i - \mathrm{Y}\Big(\omega^i_j(\mathrm{X})\Big)\mathrm{S}_i - \omega^i_k(\mathrm{Y})\omega^k_j(\mathrm{X})\mathrm{S}_i - \omega^i_j([\mathrm{X}\ \mathrm{Y}])\mathrm{S}_i\\ &= \underbrace{\mathrm{X}\Big(\omega^i_j(\mathrm{Y})\Big)\mathrm{S}_i - \mathrm{Y}\Big(\omega^i_j(\mathrm{X})\Big)\mathrm{S}_i - \omega^i_j([\mathrm{X}\ \mathrm{Y}])\mathrm{S}_i}_{\triangle} + \underbrace{\omega^i_k(\mathrm{X})\omega^k_j(\mathrm{Y})\mathrm{S}_i - \omega^i_k(\mathrm{Y})\omega^k_j(\mathrm{X})\mathrm{S}_i}_{\triangledown}\\ &= \mathrm{d}\omega^i_j(\mathrm{X}, \mathrm{Y})\mathrm{S}_j + \omega^i_k \wedge \omega^k_j(\mathrm{X}, \mathrm{Y})\mathrm{S}_i = \Big(\mathrm{d}\omega^i_j(\mathrm{X}, \mathrm{Y}) + \omega^i_k \wedge \omega^k_j(\mathrm{X}, \mathrm{Y})\Big)\mathrm{S}_i = \Omega^i_j(\mathrm{X}, \mathrm{Y})\mathrm{S}_i\end{aligned}$$

This result is valid for all X and for all Y, we conclude that $\mathrm{d}\omega^i_j + \omega^i_k \wedge \omega^k_j = \Omega^i_j$ or, in a more condensed expression, $\mathrm{d}\omega + \omega \wedge \omega = \Omega$. □

Proposition 1.8 - Bianchi identities via differential forms Let ∇ be a connection on ξ. Denote by ω and Ω the connection 1-form and the curvature 2-form of the connection ∇ respectively. Then the expression of the Bianchi identities via differential forms is

$$\mathrm{d}\Omega = \Omega \wedge \omega - \omega \wedge \Omega \tag{A.3}$$

Proof. By exterior differentiation on both sides of the equation $\Omega = \mathrm{d}\omega + \omega \wedge \omega$, gives $\mathrm{d}\Omega = \mathrm{d}^2\omega + \mathrm{d}(\omega \wedge \omega) = \mathrm{d}\omega \wedge \omega - \omega \wedge \mathrm{d}\omega = (\Omega - \omega \wedge \omega) \wedge \omega - \omega \wedge (\Omega - \omega \wedge \omega) = \Omega \wedge \omega - \omega \wedge \omega \wedge \omega - \omega \wedge \Omega + \omega \wedge \omega \wedge \omega = \Omega \wedge \omega - \omega \wedge \Omega$. □

Proposition 1.9 - Connection and curvature transformation rules Let ∇ be a connection on a vector bundle $(\mathbb{V}, \pi, \mathcal{M})$ of rank r over an m-dimensional manifold. Let $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$ be two neighborhoods of a point $\mathrm{M} \in \mathcal{M}$. Consider $\varphi_\alpha : \pi^{-1}(\mathcal{O}_\alpha) \longrightarrow \mathcal{O}_\alpha \times \mathbb{R}^r$ and $\varphi_\beta : \pi^{-1}(\mathcal{O}_\beta) \longrightarrow \mathcal{O}_\beta \times \mathcal{R}^r$. The transition map is $g_{\alpha\beta} : \mathcal{O}_\alpha \cap \mathcal{O}_\beta \longrightarrow \mathrm{GL}(n; \mathbb{R}^r)$. Denote respectively by $\omega(\alpha)$ and $\omega(\beta)$ the expressions of the connection 1-form of ∇ on $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$. Denote respectively by $\Omega(\alpha)$ and $\omega(\beta)$ and $\Omega(\beta)$ the expressions of the curvature 2-form of ∇ on $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$ respectively. Then

$$\begin{aligned} \omega(\beta) &= g^{-1}_{\alpha\beta}\mathrm{d}g_{\alpha\beta} + g^{-1}_{\alpha\beta}\omega(\alpha)g_{\alpha\beta} && \text{(A.4)} \\ \Omega(\beta) &= g^{-1}_{\alpha\beta}\Omega(\alpha)g_{\alpha\beta} && \text{(A.5)} \end{aligned}$$

Proof. Let $\mathrm{X} = (\mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_m)$ and $\mathrm{Y} = (\mathrm{Y}_1, \mathrm{Y}_2, \ldots, \mathrm{Y}_m))$ be two moving frames on $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$ respectively. We have $\mathrm{Y} = g_{\alpha\beta}\mathrm{X}$ which is a condensed way to write $(\mathrm{Y}_1, \mathrm{Y}_2, \ldots \mathrm{Y}_m) = (\mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_m)g_{\alpha\beta})$ where $g_{\alpha\beta} : \mathcal{O}_\alpha \cup \mathcal{O}_\beta \longrightarrow \mathrm{GL}(m; \mathbb{R})$. Hence

$$\mathrm{Y}_j = g^i_j \mathrm{X}_i \tag{A.6}$$

where the m^2 functions g^i_j are the components of the matrix$g_{\alpha\beta}$. Let ξ be a tangent vector field. By applying ∇_ξ to (A.6), we obtain

$$\underbrace{\nabla_\xi \mathrm{Y}_j}_{\triangleleft} = \underbrace{\nabla_\xi (g^i_j \mathrm{X}_i)}_{\triangleright} \tag{A.7}$$

On one hand, $(\triangleleft)$:$\nabla_\xi \mathrm{Y}_j = \omega(\beta)^k_j \mathrm{Y}_k = \omega(\beta)^k_j (g^i_k \mathrm{X}_i) = \omega(\beta)^k_j g^i_k \mathrm{X}_i$. On the other hand, $(\triangleright)$: $\nabla_\xi (g^i_j \mathrm{X}_i) = \xi(g^i_j)\mathrm{X}_i + g^i_j \omega(\alpha)^k_i \mathrm{X}_k = \mathrm{d}g^i_j(\xi)\mathrm{X}_i + g^k_j \omega(\alpha)^i_k \mathrm{X}_i$. By replacing the expression of the terms $(\triangleleft)$ and $(\triangleright)$ in A.7, we obtain

$$\omega(\beta)^k_j g^i_k \mathrm{X}_i = \mathrm{d}g^i_j(\xi)\mathrm{X}_i + g^k_j \omega(\alpha)^i_k \mathrm{X}_i \tag{A.8}$$

and hence

$$\omega(\beta)^k_j g^i_k \mathrm{X}_i = \mathrm{d}g^i_j(\xi)\mathrm{X}_i + g^k_j \omega(\alpha)^i_{ik} \mathrm{X}_i \tag{A.9}$$

These expressions are valid for all ξ. Consequently $g_{\alpha\beta}\omega(\beta) = \mathrm{d}g_{\alpha\beta} + \omega(\alpha)g_\alpha$, and by multiplying both sides by $g^{-1}_{\alpha\beta}$, we obtain

$$\omega(\beta) = g^{-1}_{\alpha\beta} dg_{\alpha\beta} + g^{-1}_{\alpha\beta}\omega(\alpha)g_{\alpha\beta} \tag{A.10}$$

The expression of the connection 1-form in another coordinate system is established, we will deduce the new expression of the curvature 2-form. Since $\Omega(\beta) = \mathrm{d}\omega(\beta) + \omega(\beta) \wedge \omega(\beta)$ with $\omega(\beta) = g_{\alpha\beta}^{-1}\mathrm{d}g_{\alpha\beta} + g_{\alpha\beta}^{-1}\omega(\alpha)g_{\alpha\beta}$. [1] on one hand $\mathrm{d}\omega(\beta)$ is equal to

$$\begin{aligned}\mathrm{d}(g_{\alpha\beta}^{-1}\mathrm{d}g_{\alpha\beta} + g_{\alpha\beta}^{-1}\omega(\alpha)g_{\alpha\beta}) = {} & \underbrace{\mathrm{d}g_{\alpha\beta}^{-1} \wedge g_{\alpha\beta}}_{\triangle_1} + \underbrace{g_{\alpha\beta}^{-1} \wedge \mathrm{d}(\mathrm{d}g_{\alpha\beta})}_{=0 \text{ since } \mathrm{d}^2=0} + \underbrace{\mathrm{d}g_{\alpha\beta}^{-1} \wedge \omega(\alpha)g_{\alpha\beta}}_{\nabla_1} + \underbrace{g_{\alpha\beta}^{-1}\mathrm{d}\omega(\alpha)g_{\alpha\beta}}_{\Diamond_1} \\ & - \underbrace{g_{\alpha\beta}^{-1}\omega(\alpha)\mathrm{d}g_{\alpha\beta}}_{\Box_1}\end{aligned}$$

On another hand,

$$\begin{aligned}\omega(\beta) \wedge \omega(\beta) = \Big(g_{\alpha\beta}^{-1}\mathrm{d}g_{\alpha\beta} + g_{\alpha\beta}^{-1}\omega(\alpha)g_{\alpha\beta}\Big) \wedge \Big(g_{\alpha\beta}^{-1}\mathrm{d}g_{\alpha\beta} + g_{\alpha\beta}^{-1}\omega(\alpha)g_{\alpha\beta}\Big) = \underbrace{g_{\alpha\beta}^{-1}\mathrm{d}g_{\alpha\beta} \wedge g_{\alpha\beta}^{-1}\mathrm{d}g_{\alpha\beta}}_{\triangle_2} \\ + \underbrace{g_{\alpha\beta}^{-1}\mathrm{d}g_{\alpha\beta} \wedge g_{\alpha\beta}^{-1}\omega(\alpha)g_{\alpha\beta}}_{\nabla_2} + \underbrace{g_{\alpha\beta}^{-1}\omega(\alpha)g_{\alpha\beta} \wedge g_{\alpha\beta}^{-1}dg_{\alpha\beta}}_{\Box_2} + \underbrace{g_{\alpha\beta}^{-1}\omega(\alpha)g_{\alpha\beta} \wedge g_{\alpha\beta}^{-1}\omega(\alpha)g_{\alpha\beta}}_{\Diamond_2}\end{aligned}$$

However, $g_{\alpha\beta}^{-1}g_{\alpha\beta} = \mathrm{Id}$ and hence $\mathrm{d}(g_{\alpha\beta}^{-1}g_{\alpha\beta}) = \mathrm{d}(g_{\alpha\beta})g_{\alpha\beta} + g_{\alpha\beta}^{-1}\mathrm{d}g_{\alpha\beta} = \mathrm{d}(\mathrm{Id}) = 0$. We then have $\mathrm{d}g_{\alpha\beta}^{-1}g_{\alpha\beta} = -g_{\alpha\beta}^{-1}\mathrm{d}g_{\alpha\beta}$, and by composing the two right sides by $g_{\alpha\beta}^{-1}$ we conclude the following result:

$$\mathrm{d}g_{\alpha\beta}^{-1} = -g_{\alpha\beta}^{-1}\mathrm{d}(g_{\alpha\beta})g_{\alpha\beta}^{-1} \tag{A.11}$$

In the sum $\mathrm{d}\omega(\beta) + \omega(\beta) \wedge \omega(\beta)$, we will show that the terms $(\triangle_1 + \triangle_2), (\nabla_1 + \nabla_2)$ and $(\Box_1 + \Box_2)$ vanish and the only remaining terms are $(\Diamond_1)$ and $(\Diamond_2)$.

$(\triangle_1 + \triangle_2) = \mathrm{d}g_{\alpha\beta}^{-1} \wedge \mathrm{d}g_{\alpha\beta} + g_{\alpha\beta}^{-1}(\mathrm{d}g_{\alpha\beta})g_{\alpha\beta}^{-1} \wedge \mathrm{d}g_{\alpha\beta} = \mathrm{d}g_{\alpha\beta}^{-1} \wedge \mathrm{d}g_{\alpha\beta} - \mathrm{d}g_{\alpha\beta}^{-1} \wedge \mathrm{d}g_{\alpha\beta} = 0.$

$(\nabla_1 + \nabla_2) = -g_{\alpha\beta}^{-1}\mathrm{d}(g_{\alpha\beta})g_{\alpha\beta}^{-1} \wedge \omega(\alpha)g_{\alpha\beta} + g_{\alpha\beta}^{-1}\mathrm{d}g_{\alpha\beta} \wedge g_{\alpha\beta}^{-1}\omega(\alpha)g_{\alpha\beta} = -g_{\alpha\beta}^{-1}\mathrm{d}(g_{\alpha\beta})g_{\alpha\beta}^{-1} \wedge \omega(\alpha)g_{\alpha\beta} + g_{\alpha\beta}^{-1}\mathrm{d}g_{\alpha\beta}g_{\alpha\beta}^{-1} \wedge \omega(\alpha)g_{\alpha\beta} = 0.$

Finally, $(\Box_1 + \Box_2) = -g_{\alpha\beta}^{-1}\omega(\alpha)\mathrm{d}g_{\alpha\beta} + g_{\alpha\beta}^{-1}\omega(\alpha)g_{\alpha\beta}g_{\alpha\beta}^{-1}\mathrm{d}g_{\alpha\beta} = -g_{\alpha\beta}^{-1}\omega(\alpha)\mathrm{d}g_{\alpha\beta} + g_{\alpha\beta}^{-1}\omega(\alpha)\mathrm{d}g_{\alpha\beta} = 0.$

The term $(\Diamond_2) = g_{\alpha\beta}^{-1}\omega(\alpha)g_{\alpha\beta}g_{\alpha\beta}^{-1} \wedge \omega(\alpha)g_{\alpha\beta} = g_{\alpha\beta}^{-1}\omega(\alpha) \wedge \omega(\alpha)g_{\alpha\beta}$ (because $g_{\alpha\beta}$, which is a 0-form, commutes with a differential p-form). Therefore

$$\begin{aligned}\Omega(\beta) = \mathrm{d}\omega(\beta) + \omega(\beta) \wedge \omega(\beta) &= g_{\alpha\beta}^{-1}\mathrm{d}\omega(\alpha)g_{\alpha\beta} + g_{\alpha\beta}^{-1}\omega(\alpha) \wedge \omega(\alpha)g_{\alpha\beta} \\ &= g_{\alpha\beta}^{-1}\Big(\mathrm{d}\omega(\alpha) + \omega(\alpha) \wedge \omega(\alpha)\Big)g_{\alpha\beta} = g_{\alpha\beta}^{-1}\Omega(\alpha)g_{\alpha\beta}.\end{aligned}$$

□

Proposition 1.11 - Connection and curvature forms of a metric connection Let $\mathrm{S} = (\mathrm{S}_1, \mathrm{S}_2, \ldots, \mathrm{S}_n)$ be an orthonormal moving frame with respect to g, i.e. $g_p(\mathrm{S}_i, \mathrm{S}_j) = \delta_{ij}$ for all $p \in \mathcal{O}$, $i, j = 1, \ldots, r$, then the matrix of 1-forms ω associated with S and the curvature matrix of 2-forms Ω are both skew-symmetric, i.e., $\omega_j^i + \omega_i^j = 0$ and $\Omega_j^i + \Omega_i^j = 0$.

[1] If φ is a differential p-form and ψ is a differential q-form, then $\mathrm{d}(\varphi \wedge \psi) = (\mathrm{d}\varphi) \wedge \psi + (-1)^p \varphi \wedge \mathrm{d}\psi$.

Proof. Let ∇ be a metric connection, $\mathrm{X} \in \Gamma(\mathrm{T}\mathcal{M})$ and $\mathrm{S} = (\mathrm{S}_1, \mathrm{S}_2, \ldots, \mathrm{S}_n)$ an orthonormal frame on $\mathbb{V}$. Thus, $\nabla_{\mathrm{X}}\Big(g(\mathrm{S}_i, \mathrm{S}_j)\Big) = g(\nabla_{\mathrm{X}}\mathrm{S}_i, \mathrm{S}_j) + g(\mathrm{S}_i, \nabla_{\mathrm{X}}\mathrm{S}_j)$. Since $g(\mathrm{S}_i, \mathrm{S}_j) = \delta_{ij}$, then $\nabla_{\mathrm{X}}\Big(g(\mathrm{S}_i, \mathrm{S}_j)\Big) = 0$, and hence

$$\begin{aligned}\nabla_{\mathrm{X}}\Big(g(\mathrm{S}_i, \mathrm{S}_j)\Big) &= g\Big(\omega_i^k(\mathrm{X})\mathrm{S}_k, \mathrm{S}_j\Big) + g\Big(\mathrm{S}_i, \omega_j^k(\mathrm{X})\mathrm{S}_k\Big) = g(\omega_i^k(\mathrm{X})\mathrm{S}_k, \mathrm{S}_j) + g(\mathrm{S}_i, \omega_j^k(\mathrm{X})\mathrm{S}_k) \\ &= \omega_i^k(\mathrm{X}) \underbrace{g(\mathrm{S}_k, \mathrm{S}_j)}_{\delta_{kj}} + \omega_j^k(\mathrm{X}) \underbrace{g(\mathrm{S}_i, \mathrm{S}_k)}_{=\delta_{ik}} = \omega_{ji}(\mathrm{X}) + \omega_{ij}(\mathrm{X}) = 0 \quad \text{for all } \mathrm{X} \in \Gamma(\mathrm{T}\mathcal{M}).\end{aligned}$$

Consequently, $\omega_j^i + \omega_i^j = 0$. The matrix of differential 1-forms ω is then skew-symmetric in an g-orthonormal moving frame. From Cartan's second-structure equation,

$$\Omega_j^i = \mathrm{d}\omega_j^i + \omega_k^i \wedge \omega_j^k = -\mathrm{d}\omega_i^j + \omega_i^k \wedge \omega_k^j = -\mathrm{d}\omega_i^j - \omega_k^j \wedge \omega_i^k = -\Big(\mathrm{d}\omega_i^j + \omega_k^j \wedge \omega_i^k\Big) = -\Omega_i^j$$

which concludes the skew-symmetry of the curvature 2-form of the connection ∇. Another way to express this result is that metric connections and their curvatures in an orthonormal moving frame are $\mathfrak{o}(n)$-valued differential forms rather than just being $\mathfrak{gl}(n)$-valued differential forms. □

Proposition 1.14 - Torsion transformation rule Let ∇ be a connection on an m-dimensional Riemannian manifold $(\mathcal{M}, g)$. Let $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$ be two neighborhoods of a point $\mathrm{M} \in \mathcal{M}$. Let us consider $\varphi_\alpha : \pi^{-1}(\mathcal{O}_\alpha) \longrightarrow \mathcal{O}_\alpha \times \mathbb{R}^m$ and $\varphi_\beta : \pi^{-1}(\mathcal{O}_\beta) \longrightarrow \mathcal{O}_\beta \times \mathbb{R}^m$. The transition map is then $g_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \longrightarrow \mathrm{GL}(n; \mathbb{R}^m)$. Denote by $\Theta(\alpha)$ and $\Theta(\beta)$ the expressions of the torsion 2-form on $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$ respectively. Then

$$\Theta(\beta) = g_{\alpha\beta}^{-1}\Theta(\alpha). \tag{A.12}$$

Proof. Since $\mathrm{S}(\beta) = \mathrm{S}(\alpha)g_{\alpha\beta}$, we can conclude that $\eta(\beta) = g_{\alpha\beta}^{-1}\eta(\alpha)$. Therefore,

$$\begin{aligned}\Theta(\beta) &= \mathrm{d}\eta(\beta) + \omega(\beta) \wedge \eta(\beta) = \mathrm{d}\Big(g_{\alpha\beta}^{-1}\eta(\alpha)\Big) + \Big(g_{\alpha\beta}^{-1}\mathrm{d}g_{\alpha\beta} + g_{\alpha\beta}^{-1}\omega(\alpha)g_{\alpha\beta}\Big) \\ &= \mathrm{d}g_{\alpha\beta}^{-1} \wedge \eta(\alpha) + g_{\alpha\beta}^{-1}\mathrm{d}\eta(\alpha) + g_{\alpha\beta}^{-1} \wedge \mathrm{d}g_{\alpha\beta} \wedge g_{\alpha\beta}^{-1}\eta(\alpha) + g_{\alpha\beta}^{-1}\omega(\alpha)g_{\alpha\beta} \wedge g_{\alpha\beta}^{-1}\eta(\alpha) \\ &= \underbrace{-g_{\alpha\beta}^{-1}(\mathrm{d}g_{\alpha\beta})g_{\alpha\beta}^{-1} \wedge \eta(\alpha)}_{\diamond} + g_{\alpha\beta}^{-1}\mathrm{d}\eta(\alpha) + \underbrace{g_{\alpha\beta}^{-1}(\mathrm{d}g_{\alpha\beta})g_{\alpha\beta}^{-1} \wedge \eta(\alpha)}_{\diamond} + g_{\alpha\beta}^{-1}\omega(\alpha) \underbrace{(g_{\alpha\beta})(g_{\alpha\beta}^{-1})}_{=\mathrm{I}d} \wedge \eta(\alpha) \\ &= g_{\alpha\beta}^{-1}\mathrm{d}\eta(\alpha) + g_{\alpha\beta}^{-1}\omega(\alpha) \wedge \eta(\alpha) = g_{\alpha\beta}^{-1}\Big(\mathrm{d}\eta(\alpha) + \omega(\alpha) \wedge \eta(\alpha)\Big) = g_{\alpha\beta}^{-1}\Theta(\alpha)\end{aligned}$$

□

Proposition 1.15 - Relationship between the connection, curvature and torsion Let ∇ be a connection on an m-dimensional Riemannian manifold $(\mathcal{M}, g)$. Denote by ω the connection 1-form of ∇, Ω its curvature, Θ its torsion and $\eta = (\eta^1, \ldots, \eta^m)$ a moving coframe. Then the connection, the curvature, torsion and the coframe are related by

$$\mathrm{d}\Theta + \omega \wedge \Theta = \Omega \wedge \eta. \tag{A.13}$$

Proof. By definition, $\mathrm{d}\eta + \omega \wedge \eta = \Theta$. By exterior differentiation, we obtain $\underbrace{\mathrm{d}^2\eta}_{=0} + \mathrm{d}(\omega \wedge \eta) = \mathrm{d}\Theta$, and hence

$$\mathrm{d}\Theta = \mathrm{d}\omega \wedge \eta - \omega \wedge \mathrm{d}\eta. \tag{A.14}$$

On one hand, $\mathrm{d}\omega+\omega\wedge\omega=\Omega$, and hence $\mathrm{d}\omega=\Omega-\omega\wedge\omega$. On the other hand, $\mathrm{d}\eta+\omega\wedge\eta=\Theta$, Consequently, $\mathrm{d}\eta=\Theta-\omega\wedge\eta$. Substituting in (A.14) gives,

$$\mathrm{d}\Theta=(\Omega-\omega\wedge\omega)\wedge\eta-\omega\wedge(\Theta-\omega\wedge\eta)=\Omega\wedge\eta-\omega\wedge\omega\wedge\eta-\omega\wedge\Theta+\omega\wedge\omega\wedge\eta=\Omega\wedge\eta-\omega\wedge\Theta \quad (A.15)$$

□

Lemma 1.17 - Cartan lemma Let $\mathcal{M}$ be an m-dimensional manifold. Let us consider $\omega^1,\omega^2,\dots,\omega^r$ linearly independent differential 1-forms on $\mathcal{M}$, where $r\leqslant n$ and let $\theta^1,\theta^2,\dots,\theta^r$ be the r differential form on $\mathcal{M}$ such that $\sum_{i=1}^{r}\theta^i\wedge\omega^i=0$. There then exist r^2 functions $\mathcal{C}^\infty$ on $\mathcal{M}$ h^i_j such that $\theta^i=\sum_{j=1}^{r}h^i_j\omega^j$ where $h^i_j=h^j_i$.

Proof. Let $\{\omega^\mu\}_{\mu=1,\dots,m}$ be a basis of $\mathrm{T}^*\mathcal{M}$ such that the first r terms are as in Lemma's hypothesis (we have just completed $\omega^1,\omega^2,\dots,\omega^r$ to form a basis of $\mathrm{T}^*\mathcal{M}$). θ^i are then expressed in this basis as follows :

$$\theta^i=\sum_{\nu=1}^{m}a^i_\nu\omega_\nu \qquad \text{where the } a^i_\nu\in\mathcal{C}^\infty(\mathcal{M}),\ i=1,\dots,r.$$

Therefore

$$\begin{aligned}\sum_{i=1}^{r}\theta^i\wedge\omega^i&=\sum_{i=1}^{r}\Big(\sum_{\nu=1}^{m}a^i_\nu\omega^\nu\Big)\wedge\omega^i=\sum_{i=1}^{r}\sum_{\nu=1}^{m}a^i_\nu\omega^\nu\wedge\omega^i=\sum_{i=1}^{r}\sum_{j=1}^{r}a^i_j\omega_j\wedge\omega_i+\sum_{i=1}^{r}\sum_{\alpha=r+1}^{m}a^i_\alpha\omega^\alpha\wedge\omega^i\\&=\sum_{1\leq i<j\leq r}(a^i_j-a^j_i)\omega^j\wedge\omega^i+\sum_{i=1}^{r}\sum_{\alpha=r+1}^{m}a^i_\alpha\omega^\alpha\wedge\omega^i=0\end{aligned}$$

since the ω^μ are linearly independent, the vanishing of the sum means that each coefficient vanishes, ie., $a^i_\alpha=0$ for $\alpha=r+1,\dots,m$ and $a^i_j=a^j_i$ for $i,j=1,\dots,r$. Substituting h^i_j by a^i_j, we obtain $\theta^i=\sum_{j=1}^{r}h^i_j\omega^j$ where $h_{ij}=h_{ji}$, $i,j=1,\dots,r$. □

Appendix B

Tableaux and linear Pfaffian systems

The second and last appendix is dedicated to briefly reveiwing tableaux and linear Pfaffian systems, and their applications in finding integral manifolds. The reader may refer to [IL03, BCG+91, Car71] for more examples and explanations. We show the involution of the heat equation and the Cauchy–Riemann system, and recover some geometric results as expounded in the previous chapters: conformal correspondence between two Riemannian surfaces (chapter 1, section 3) and the existence of Lagrangian manifolds of the complex space $\mathbb{C}^m$ (chapter 3).

B.1 A short review of tableaux and linear Pfaffian systems

Let $\mathbb{X}$ and $\mathbb{U}$ be two vector spaces of dimension n and s respectively. Denote by $(e_i)_{i=1,\ldots,n}$ and $(f_a)_{a=1,\ldots,s}$ the bases of $\mathbb{X}$ and $\mathbb{U}$ respectively. An element in $\mathbb{X}$ and another in $\mathbb{U}$ are expressed $x = x^i e_i$ and $u = u^a f_a$ respectively. Denote by $(x^1, \ldots, x^n)$ and by $(u^1, \ldots, u^s)$ the coordinate functions on $(\mathbb{X})$ and $\mathbb{U}$ respectively. Any first order, constant-coefficient, homogeneous system of PDE for maps $f : \mathbb{X} \longrightarrow \mathbb{U}$ is given by

$$\mathrm{B}_a^{ri} \frac{\partial u^a}{\partial x^i}, \qquad r = 1, \ldots, \mathrm{R}. \tag{B.1}$$

where the symbol relations B_a^{ri} of the system of PDEs (B.1) are constants.

Example 2.1 – Symbols of the Cauchy–Riemann system. A map $u : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, that maps (x^1, x^2) to $(u^1(x), u^2(x))$ is a solution to the Cauchy–Riemann system if $\partial u^1/\partial x^1 - \partial u^2/\partial x^2 = \partial u^1/\partial x^2 + \partial u^2/\partial x^1 = 0$. Thus, the symbol relations of the Cauchy–Riemann system are: $\mathrm{B}_1^{11} = \mathrm{B}_1^{21} = \mathrm{B}_1^{22} = -\mathrm{B}_2^{12} = 1$ and $\mathrm{B}_1^{12} = \mathrm{B}_2^{11} = \mathrm{B}_1^{21} = \mathrm{B}_{1=0}^{22}$.

The system (B.1) can be described as a subspace $\mathrm{B} \subset \mathbb{X} \otimes \mathbb{U}^*$, where

$$\mathrm{B} = \{\mathrm{B}_a^{ri} e_i \otimes f^a | 1 \leqslant r \leqslant \mathrm{R}\}. \tag{B.2}$$

B is called the space of symbol relations.

Definition 2.2 – Tableau A tableau of a first-order, constant-coefficient, homogeneous system of PDEs is a linear subspace A of $\mathbb{U} \otimes \mathbb{X}^*$ orthogonal to its symbol relations space, i.e., $\mathrm{A} = \mathrm{B}^{\perp}$.

A linear subspace A of $\mathrm{W} \otimes \mathrm{V}^*$ determines a first-order, constant-coefficient, homogeneous system of PDEs. We usually identify the vector spaces $\mathbb{X}$ and $\mathbb{U}$ with $\mathbb{R}^n$ and $\mathbb{R}^s$ and write the tableau and the symbols relations space in matrix form.

Examples 2.3 – Tableaux.

- **Frobenius systems** The tableau $\mathrm{A} = (0)$ corresponds to a completely integrable system. The equations are $\partial u^a/\partial x^i = 0$ for all i and a, and the solutions to this system are constants maps.

- **Cauchy–Riemann system** The tableau of the Cauchy–Riemann system is:

$$\mathrm{A} = \{a(f_1 \otimes e^1 + f_2 \otimes e^2) + b(-f_2 \otimes e^1 + f_1 \otimes e^2) | a, b \in \mathbb{R}\} \simeq \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} | a, b \in \mathbb{R} \right\}. \quad \text{(B.3)}$$

- When $\mathrm{A} = \mathrm{W} \otimes \mathrm{V}^*$, there are no equations and any map is a solution.

Definition 2.4 — Prolongation of a tableau Let $\mathrm{A} \subset \mathbb{U} \otimes \mathbb{X}^*$ be a tableau and k be a positive integer. The k^{th} prolongation of the tableau A is a subspace $\mathrm{A}^{(k)} := (\mathrm{A} \otimes \mathbb{X}^{*\otimes k}) \cap (\mathbb{U} \otimes \mathrm{S}^{k+1}\mathbb{X}^*)$.

Definition 2.5 — Tableau of order p A linear subspace of $\mathbb{U} \otimes \mathrm{S}^p\mathbb{X}^*$ is a tableau of order p. It determines a homogeneous constant-coefficient system of PDEs of order p for $\mathbb{U}$-valued function on $\mathbb{X}$. In particular, the $(p-1)$-st prolongation of a tableau is a tableau of order p.

Example 2.6 — Tableau of the equation a second-order PDE. Consider $\mathbb{X} = \mathbb{R}^2$ and $\mathbb{U} = \mathbb{R}$, and the second-order PDE $\partial^2\mathbf{u}/\partial\mathbf{x}\partial\mathbf{x} + \partial^2\mathbf{u}/\partial\mathbf{y}\partial\mathbf{y} = \mathbf{0}$. Thus its tableau A is a subspace of $\mathbb{R} \otimes \mathrm{S}^2((\mathbb{R}^2)^*)$ defined by:

$$\mathrm{A} = \{a f_1 \otimes (e^1 \odot e^1 - e^2 \odot e^2) + b f_1 \otimes e^1 \odot e^2 | a, b \in \mathbb{R}\} \simeq \{\begin{pmatrix} a & b & -a \end{pmatrix} | a, b \in \mathbb{R}\}. \quad \text{(B.4)}$$

Definition 2.7 — Prolongation of a tableau of order p Let A be a tableau of order p and k be a positive integer. The k^{th} prolongation of the tableau A is a subspace $\mathrm{A}^{(k)} := (\mathrm{A} \otimes \mathbb{X}^{*\otimes k}) \cap (\mathbb{U} \otimes \mathrm{S}^{p+k}\mathbb{X}^*)$.

Recall that a Pfaffian system on a manifold Σ is an exterior differential system which contains only linearly independent differential 1-forms, i.e., $\mathrm{I} = \{\theta^a\} \subset \mathrm{T}^*\Sigma$ where $a = 1, \ldots, s$. If $\Omega = \omega^1 \wedge \cdots \wedge \omega^n$ represents a condition of independence, then $(\mathrm{I}, \mathrm{J}) := \{\theta^a, \omega^i\}$ is a Pfaffian system with an independence condition.

Definition 2.8 — Linear Pfaffian system A Pfaffian system I with a condition of independence J is a linear Pfaffian system if $\mathrm{d}\theta^a \equiv 0 \bmod \mathrm{J}$ for all $a = 1, \ldots, s$.

Example 2.9 — Canonical contact system of system of PDE. Any system of PDEs can be expressed as the pull-back of the canonical contact system on a jet space and is thus a linear Pfaffian system. For instance, consider $\mathrm{J}^1(\mathbb{R}^2, \mathbb{R}^2)$ with the coordiantes $\{x^1, x^2, u^1, u^2, p_1, q_1, p_2, q_2\}$. The canonical contact system on $\mathrm{J}^1(\mathbb{R}^2, \mathbb{R}^2)$ is:

$$\begin{cases} \theta^1 := \mathrm{d}u^1 - p_1\mathrm{d}x^1 - q_1\mathrm{d}x^2 \\ \theta^2 := \mathrm{d}u^2 - p_2\mathrm{d}x^1 - q_2\mathrm{d}x^2. \end{cases} \quad \text{(B.5)}$$

One can check that $\mathrm{d}\theta^1 \equiv 0$ and $\mathrm{d}\theta^2 \equiv 0 \bmod \{\mathrm{d}x^1, \mathrm{d}x^2, \theta^1, \theta^2\}$, and thus it is a linear Pfaffian system. The Cauchy–Riemann system is determined by $p_1 - q_2 = p_2 + q_1 = 0$ on $\mathrm{J}^1(\mathbb{R}^2, \mathbb{R}^2)$. The Cauchy–Riemann system is then equivalent to the pull-back of (B.5) on the manifold Σ^6 defined by the vanishing of the functions $\mathrm{F}_1 := p_1 - q_2$ and $\mathrm{F} := p_2 + q_1$. Note that the pull-back of the canonical system on any manifold $\Sigma \subset \mathrm{J}^1(\mathbb{R}^2, \mathbb{R}^2)$ is also a linear Pfaffian system.

Proposition 2.10 Let I be a linear Pfaffian system with an independence condition J. Let π^ε, $\varepsilon = 1, \ldots, \dim\Sigma - n - s$, be a collection of 1-forms such that $T^*\Sigma$ is spanned by $\theta^a, \omega^i, \pi^\varepsilon$. Then there exist functions $A^a_{\varepsilon i}$ and T^a_{ij} defined on Σ such that

$$d\theta^a \equiv A^a_{\varepsilon i}\pi^\varepsilon \wedge \omega^i + T^a_{ij}\omega^i \wedge \omega^j \mod I. \tag{B.6}$$

The functions T^a_{ij} are called the apparent torsion of the linear Pfaffian system. If we replace the differential 1-forms π^ε by $\tilde{\pi}^\varepsilon = \pi^\varepsilon + M^\varepsilon_\delta\theta^\delta$, where (M^ε_δ) is an invertible matrix, the apparent torsion in the new coframe remains unchanged, i.e., $\tilde{T}^a_{ij} = T^a_{ij}$. It appears that the only possible way to absorb the apparent torsion is to change the complement of J/I. For instance, if it is possible to choose a matrix (M^ε_j) such that $\tilde{T}^a_{ij} = A^a_{\varepsilon i}M^\varepsilon_j + T^a_{ij} = 0$ with $\tilde{\pi} = \pi + M^\varepsilon_j\omega^j$, then the apparent torsion of the linear Pfaffian system is said to be absorbable, and otherwise, there is torsion. We denote by $[T] = 0$ an apparent torsion which is absorbable.

Given a tableau $A \subset \mathbb{U} \otimes \mathbb{X}^*$ expressed in terms of bases $e = (e^1, \ldots, e^n)$ of $\mathbb{X}^*$ and $u = (f_1, \ldots, f_s)$ of $\mathbb{U}$ and let $s_1(e), \ldots, s_n(e)$ be defined by:

$$\begin{aligned} s_1(e) &= \#\text{ of independent entries in the first column of A,} \\ s_1(e) + s_2(e) &= \#\text{ of independent entries in the first 2 columns of A,} \\ &\vdots \\ s_1(e) + \cdots + s_n(e) &= \#\text{ of independent entries in A.} \end{aligned}$$

In other words, $s_k(e)$ is the number of new independent entries in the k^{th} column. The characters $s_k(e)$ do not depend on the choice of the basis of $\mathbb{U}$ but only on the flag $F = (F_0, \ldots, F_n)$ of subspaces in $\mathbb{X}^*$ induced by e, where $F_j = \text{span}\{e^{j+1}, \ldots, e^n\}$ with $F_0 = \mathbb{X}^*$ and $F_n = (0)$. Define

$$A_k(F) = (\mathbb{U} \otimes F_k) \cap A$$

and hence

$$\dim A_k(F) = s_{k+1}(F) + \cdots + s_n(F) \tag{B.7}$$

Definition 2.11 – Characters of a tableau Let $A \subset \mathbb{U} \otimes \mathbb{X}^*$ be a tableau. The characters s_k, for $k = 1, \ldots, n$, are defined by:

$$\begin{aligned} s_1(A) &= \max\{s_1(F)|\text{ all flags}\}, \\ s_2(A) &= \max\{s_2(F)|\text{ flags with } s_1(F) = s_1(A)\}, \\ &\vdots \\ s_n(A) &= \max\{s_1(F)|\text{ flags with} s_1(F) = s_1(A), \ldots, s_{n-1}(F) = s_{n-1}(A)\}. \end{aligned}$$

Remark 2.12 – The characters $\mathbf{s_k}$ vs $\mathbf{C_k}$. The characters s_k are related to the characters C_k defined in chapter 2 by:

$$s_k = C_k - C_{k-1} \quad \text{for } 1 \leqslant n-1, \qquad \text{and} \qquad s_n = \text{codim}E - C_{n-1}. \tag{B.8}$$

Example 2.13 — Cauchy–Riemann system-continued. Recall that the tableau of the Cauchy–Riemann system is of the form (B.3). Thus, the first character of the tableau is $s_1 = 2$ and $s_2 = 0$. Using the pull-back on $\Sigma^6 = \{p_1 - q_2 = p_2 + q_1 = 0\}$ of the canonical contact system of $\mathrm{J}^1(\mathbb{R}^2, \mathbb{R}^2)$, we find (fortunately) the same characters: on Σ^6, $\mathrm{d}p_1 = \mathrm{d}q_2$ and $\mathrm{d}p_2 = -\mathrm{d}q_1$.

$$\mathrm{d}\begin{pmatrix} \theta^1 \\ \theta^2 \end{pmatrix} \equiv -\underbrace{\begin{pmatrix} \mathrm{d}p_1 & -\mathrm{d}q_1 \\ \mathrm{d}q_1 & \mathrm{d}p_1 \end{pmatrix}}_{\mathrm{A}} \wedge \begin{pmatrix} \mathrm{d}x^1 \\ \mathrm{d}x^2 \end{pmatrix} \tag{B.9}$$

Note that the Cauchy–Riemann system has no torsion.

The properties of the tableau first prolongation leads to

Proposition-Definition 2.14 — Involutive tableau Let $\mathrm{A} \subset \mathbb{U} \otimes \mathbb{X}^*$ be a tableau. Then

$$\dim \mathrm{A}^{(1)} \leqslant s_1 + 2s_2 + 3s_3 + \cdots + ns_n. \tag{B.10}$$

If equality holds, then the tableau A is said to be involutive.

Example 2.15 — Cauchy–Riemann system-continued. The dimension of the first-prologation of the Cauchy–Riemann system's tabelau is $\dim \mathrm{A}^{(1)} = 2$ which is equal to $s_1 + 2s_2$. Hence, the tableau is involutive.

Theorem 2.16 — Cartan–Kähler theorem for linear Pfaffian systems Let (I, J) be an analytic linear Pfaffian system on a manifold Σ, let $x \in$ be a point and let $\mathcal{O}$ be a neighborhood containing x, such that for all $y \in \mathcal{O}$, there is no torsion $[\mathrm{T}]_y$ and the tableau A_y is involutive. Then, solving a series of Cauchy problems yields analytic integral manifolds to (I, J) passing through x that depend on s_l functions of l variables, where s_l is the character of the system[1].

If the linear Pfaffian system, or more generally an exterior differential system, is not involutive, then we prolong and restart again. Indeed, given an exterior differential system defined on a manifold $\mathcal{M}$, the Cartan-Kuranishi prolongation theorem [Kur57] says that after a finite number of prolongations, the system is either in involution (admits at least one 'large' integral manifold), or is impossible. Reproduced in figure B.1 is the formal algorithm in [IL03] for finding whether or not a linear Pfaffian system posses integral manifolds.

B.2 Applications

Presented here are some applications to the method of tableaux and linear Pfaffian system. We start by a partial differential equation, then we treat geometric problems (the conformal embedding of Riemannian manifolds) and present another way to proof the existence of Lagrangian manifolds of symplectic and complex spaces.

B.2.1 Heat equation

Let u be a function on $\mathbb{R}^2$. We are looking for solutions to the heat equation

$$\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial y \partial y} = 0. \tag{B.11}$$

[1] The last non-vanishing character of the tableau.

Let (x, y, u, p, q, r, s, t) be a coordinate system on the 2-jet space $\mathrm{J}^2(\mathbb{R}^2, \mathbb{R})$.Consequently, the heat equation is equivalent to the pull-back by $\iota : \Sigma^7 \longrightarrow \mathrm{J}^2(\mathbb{R}^2, \mathbb{R})$ on Σ^7 of the canonical contact system

$$\left\{\begin{array}{lcl} \theta^0 & = & \mathrm{d}u - p\mathrm{d}x - q\mathrm{d}y \\ \theta^1 & = & \mathrm{d}p - r\mathrm{d}x - s\mathrm{d}y \\ \theta^2 & = & \mathrm{d}u - s\mathrm{d}x - t\mathrm{d}y \end{array}\right. \tag{B.12}$$

where Σ^7 is defined by the equation $p - t = 0$. The pull-back of the forms θ^0, θ^1 and θ^2 are denoted by the same symbol. On Σ^7, $\mathrm{d}p = \mathrm{d}t$, and thus

$$\mathrm{d}\begin{pmatrix} \theta^0 \\ \theta^1 \\ \theta^2 \end{pmatrix} \equiv -\underbrace{\begin{pmatrix} 0 & 0 \\ \mathrm{d}r & \mathrm{d}s \\ \mathrm{d}s & \mathrm{d}p \end{pmatrix}}_{:=\mathrm{A}} \wedge \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix}. \tag{B.13}$$

The linear Pfaffian system has no torsion and $s_1(\mathrm{A}) = 2$ and $s_2(\mathrm{A}) = 1$. Besides, $\mathrm{dim}\mathrm{A}^{(1)} = 4$. Hence, the linear Pfaffian system associated to the heat equation is in involution since it passes the Cartan test, i.e., $s_1 + 2s_2 = \mathrm{dim}\mathrm{A}^{(1)} = 4$. Solutions to the heat equation depend upon two functions of one variable and one function of two variables.

B.2.2 CONFORMAL EMBEDDINGS

Definition 2.17 – Conformal map Let $(\mathcal{M}^m, g)$ and $(\mathcal{N}^n, h)$ be two real analytic Riemannian manifolds of dimension m and n respectively. A map $u : \mathcal{M}^m \longrightarrow \mathcal{N}^n$ is conformal if there exists a non-vanishing function λ on $\mathcal{M}^2$ such that $\mathrm{u}^*(h) = \lambda^2 g$.

The following result by Jacobowitz and Moore [JM73] provides a condition for the local existence of a conformal map between real analytic Riemannian manifolds. They gave two different proofs for the following result, one based on Janet's method and the other on Cartan's method.

Theorem 2.18 – Local conformal embedding of Riemannian manifolds Let $(\mathcal{M}^m, g)$ and $(\mathcal{N}^n, h)$ be two real analytic Riemannian manifolds of dimension m and n respectively, $m \geqslant 2$ and $n \geqslant m(m+1)/2 - 1$. If M is a point of $\mathcal{M}^m$, there exists an open neighborhood $\mathcal{O}$ of M in $\mathcal{M}^m$ which can be conformally embedded in $\mathcal{N}^n$.

In particular, any two real analytic Riemannian surfaces are locally conformal, and any real analytic Riemannian surface is locally conformal to the Euclidean space $(\mathbb{R}^2, \langle,\rangle_{\mathbb{R}^2})$. Using tableaux and linear Pfaffian systems, we can prove the local conformal embedding of surfaces.

Proof. Let $(\mathcal{M}^2, g)$ and $(\mathcal{N}^2, h)$ be two real analytic Riemannian surfaces. Denote by (η^1, η^2) the g-orthonormal coframe of $(\mathcal{M}^2, g)$ on which the metric g is diagonal, and denote by (ω^1, ω^2) the h-orthonormal coframe of $(\mathcal{N}^2, h)$ on which the metric h is diagonal[2], i.e.,

$$g = \eta^1 \otimes \eta^1 + \eta^2 \otimes \eta^2 \qquad \text{and} \qquad h = \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2. \tag{B.14}$$

Cartan's structure equations for $(\mathcal{M}^2, g)$ and $(\mathcal{N}^2, h)$ are:

$$\left\{\begin{array}{lcl} \mathrm{d}\eta^1 + \eta^1_2 \wedge \eta^2 & = & 0 \\ \mathrm{d}\eta^2 - \eta^1_2 \wedge \eta^1 & = & 0 \\ \mathrm{d}\eta^1_2 - \mathcal{K}_{\mathcal{M}}\eta^1 \wedge \eta^2 & = & 0 \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{lcl} \mathrm{d}\omega^1 + \omega^1_2 \wedge \omega^2 & = & 0 \\ \mathrm{d}\omega^2 - \omega^1_2 \wedge \omega^1 & = & 0 \\ \mathrm{d}\omega^1_2 - \mathcal{K}_{\mathcal{N}}\omega^1 \wedge \omega^2 & = & 0 \end{array}\right. \tag{B.15}$$

[2] As in Cartan–Janet theorem's proof.

where η_2^1 and ω_2^1 are the non-vanishing terms of the connection 1-form of the Levi-Civita on $(\mathcal{M}^2, g)$and $(\mathcal{N}^2, h)$ respectively.

By definition, if the two Riemannian surfaces are conformal, then there exists a non-vanishing function λ on $\mathcal{M}^2$ such that the pull-back of h is $\lambda^2.g$. Consider then in $\mathcal{M}^2 \times \mathcal{N}^2$ the following Pfaffian system[3]

$$\mathrm{I} = \{\omega^1 - \lambda\eta^1, \omega^2 - \lambda\eta^2\} \tag{B.16}$$

with the independence condition $\mathrm{J} = \{\eta^1, \eta^2, \omega^1, \omega^2\}$. One can easily check that (I, J) is a linear Pfaffian system. Using Cartan's structure equations, we have:

$$\mathrm{d}\begin{pmatrix} \omega^1 - \lambda\eta^1 \\ \omega^2 - \lambda\eta^2 \end{pmatrix} \equiv -\underbrace{\begin{pmatrix} \mathrm{d}\lambda/\lambda & (\omega_2^1 - \eta_2^1) \\ -(\omega_2^1 - \eta_2^1) & \mathrm{d}\lambda/\lambda \end{pmatrix}}_{:=\mathrm{A}} \wedge \begin{pmatrix} \eta^1 \\ \eta^2 \end{pmatrix} \quad \mathrm{mod}\ \mathrm{I} \tag{B.17}$$

The linear Pfaffian system (I, J) has non torsion. On one hand, $s_1(\mathrm{A}) = 2$ and $s_2(\mathrm{A}) = 0$. On the other hand, $\mathrm{dim}\mathrm{A}^{(1)} = 2$. Thus, the linear Pfaffian system (I, J) passes the Cartan test, i.e., $s_1 + 2s_2 = \mathrm{dim}\mathrm{A}^{(1)} = 2$, and hence, by the Cartan–Kähler theorem, there exists a local conformal embedding of $\mathcal{M}^2$ in $\mathcal{N}^2$. □

B.2.3 LAGRANGIAN MANIFOLDS IN $\mathbb{C}^m$

Let $\mathbb{C}^m$ be a complex space of complex dimension m. As mentioned in the chapter 3, dealing with the isometric Lagrangian embedding of surfaces, the complex structure J and a Euclidean metric induce a symplectic structure. Thus, $(\mathbb{C}, \mathrm{J})$ can be viewed as $(\mathbb{R}^{2m}, \omega, \langle, \rangle_{\mathbb{R}^{2m}})$, where $\omega(.,.) = -\langle ., \mathrm{J}.\rangle_{\mathbb{R}^{2m}}$. Consider for instance on $\mathbb{C}^m$ the complex structure J which induces $\omega = \mathrm{d}x^1 \wedge \mathrm{d}x^m + \ldots \mathrm{d}x^m \wedge \mathrm{d}x^{2m}$. Lagrangian manifolds of $\mathbb{C}^m$ are the integral manifolds of ω, and, as explained in the isometric Lagrangian embedding (chapter 3), are obtained by considering on the unitary frame bundle $\mathcal{F}(\mathbb{C}^m)$, the exterior differential system $\mathrm{I} = \{\omega^m, \ldots, \omega^{2m}\}$ with the condition of independence $\Omega = \omega^1 \wedge \cdots \wedge \omega^m$. Note that (I, J) is a linear Pfaffian system. Matrices in the Lie algebra of $\mathfrak{u}(m)$ of $\mathrm{U}(m) = \mathrm{SO}(2m) \cap \mathrm{S}p(\mathbb{R}^{2m}, \omega)$, for $m = 2$ and 3, have the form

$$\begin{pmatrix} 0 & -\xi & -\alpha & \eta \\ \xi & 0 & -\eta & -\beta \\ \alpha & \eta & 0 & -\xi \\ \eta & \beta & \xi & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\xi^1 & -\xi^2 & -\alpha & -\eta^1 & -\eta^2 \\ \xi^1 & 0 & -\xi^3 & -\eta^2 & -\beta & -\eta^3 \\ \xi^2 & \xi^3 & 0 & -\eta^2 & -\eta^3 & -\gamma \\ \alpha & \eta^1 & \eta^2 & 0 & -\xi^1 & -\xi^2 \\ \eta^1 & \beta & \eta^3 & \xi^1 & 0 & -\xi^3 \\ \eta^2 & \eta^3 & \gamma & \xi^2 & \xi^3 & 0 \end{pmatrix} \tag{B.18}$$

respectively. For m arbitrary, a matrix in $\mathfrak{u}(m)$ has the form

$$\begin{pmatrix} \mathrm{A} & -\mathrm{S} \\ \mathrm{S} & \mathrm{A} \end{pmatrix} \tag{B.19}$$

where $\mathrm{S} \in \mathrm{M}_m(\mathbb{R})$ is symmetric and $\mathrm{A} \in \mathrm{M}_m(\mathbb{R})$ is skew-symmetric. We need to check the involutivity of the linear Pfaffian system (I, J). The general case, i.e., for an arbitrary complex dimension m, can be done. However, for more clarity, the case $m = 2$ and $m = 3$ are expounded before the general case.

[3]We denote by the same symbol the pull-back of the forms by the natural projections on $\mathcal{M}^2$ and $\mathcal{N}^2$.

Lagrangian manifolds of $\mathbb{C}^2$

$$\mathrm{d}\begin{pmatrix} \omega^3 \\ \omega^4 \end{pmatrix} \equiv -\begin{pmatrix} \omega_1^3 & \omega_2^3 \\ \omega_1^4 & \omega_2^4 \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} \tag{B.20}$$

The Levi-Civita connection on $\mathbb{C}^2$ is a $\mathfrak{u}(2)$-valued differential form. Hence, $\omega_1^4 = \omega_2^3$ and the character of the tableau are $s_1 = 2$ and $s_2 = 1$. By the Cartan lemma, $\mathrm{dim}\mathrm{A}^{(1)} = 4$. The tableau is then involutive.

Lagrangian manifolds in $\mathbb{C}^3$

$$\mathrm{d}\begin{pmatrix} \omega^4 \\ \omega^5 \\ \omega^6 \end{pmatrix} \equiv -\begin{pmatrix} \omega_1^4 & \omega_2^4 & \omega_3^4 \\ \omega_1^5 & \omega_2^5 & \omega_3^5 \\ \omega_1^6 & \omega_2^6 & \omega_3^6 \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} \tag{B.21}$$

The Levi-Civita connection on $\mathbb{C}^3$ is a $\mathfrak{u}(3)$-valued differential 1-form. Hence, $\omega_1^5 = \omega_2^4$, $\omega_1^6 = \omega_3^4$, $\omega_2^6 = \omega_3^5$ and the character of the tableau are $s_1 = 3$, $s_2 = 2$ and $s_3 = 1$. The dimension of the first prolongation $\mathrm{A}^{(1)}$ is 10.

Lagrangian manifolds of $\mathbb{C}^m$

$$\mathrm{d}\begin{pmatrix} \omega^{m+1} \\ \vdots \\ \omega^{2m} \end{pmatrix} \equiv -\underbrace{\begin{pmatrix} \omega_1^{m+1} & \dots & \omega_1^{m+1} \\ \vdots & & \vdots \\ \omega_m^{2m} & \dots & \omega_m^{2m} \end{pmatrix}}_{:=\mathrm{A}} \wedge \begin{pmatrix} \omega^1 \\ \vdots \\ \omega^m \end{pmatrix} \tag{B.22}$$

The Levi-Civita connection on $\mathbb{C}^3$ is a $\mathfrak{u}(m)$-valued differential 1-form. Hence, the tableau A is symmetric and the characters of the tableau are:

$$s_i = m - i + 1 \qquad \text{for all} \quad i = 1, \dots, m. \tag{B.23}$$

The Cartan lemma implies that there are $\omega_j^{m+i} = h_{jk}^{m+i}\omega^k$, for all $i, j = 1, \dots, m$, where $h_{jk} = h_{kj}$. There should be $m \times m(m+1)/2$ functions. However, the matrix of 1-forms (ω_j^{n+i}) is symmetric, and thus there are $\sum_{k=1}^m k(k+1)/2$ and hence

$$\mathrm{dim}\mathrm{A}^{(1)} = m(m+1)(2m+1)/6. \tag{B.24}$$

Since $s_1 + 2s_2 + \dots, ms_m = \sum_{i=1}^{m} m - i + 1 = \mathrm{dim}\mathrm{A}^{(1)}$, the tableau A is involutive and thus there exist Lagrangian manifolds in $\mathbb{C}^m$.

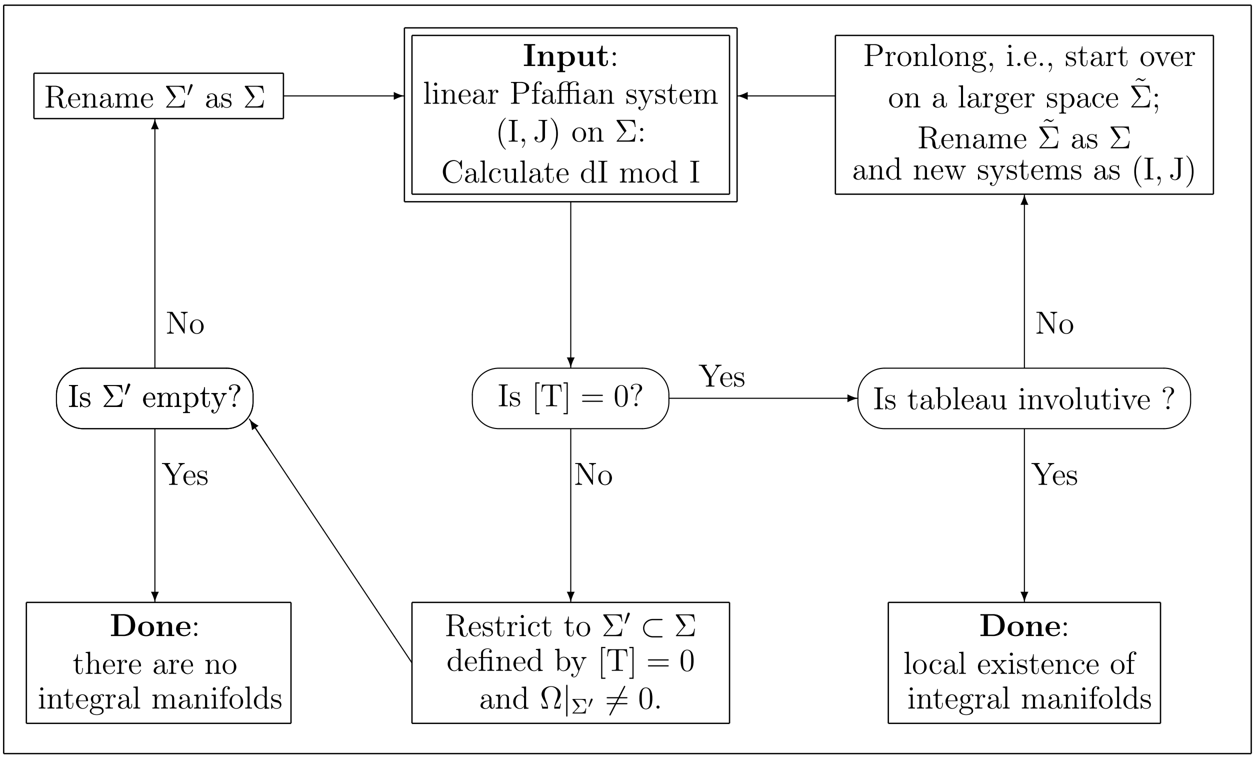

Figure B.1: Linear Pffafian system algorithm

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